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Observers for Linear Systems

J. O'Reilly

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OBSERVERS FOR LINEAR SYSTEMS

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Preface

In 1963, David G. Luenberger initiated the theory of observers for the state reconstruction of linear dynamical systems. Since then, owing to its utility and its intimate connection with fundamental system concepts, observer theory continues to be a fruitful area of research and has been substantially developed in many different directions. In view of this, the observer has come to take its pride of place in linear multivariable control alongside the optimal linear regulator and the Kalman filter. Notwithstanding the importance of the observer and its attendant vast literature, there exists, at the time of writing, no single text dedicated to the subject.

My aim, in writing this monograph, has been to remedy this omission by presenting a comprehensive and unified theory of observers for continuous-time and discrete-time linear systems. The book is intended for post-graduate students and researchers specializing in control systems, now a core subject in a number of disciplines. Forming, as it does, a self-contained volume it should also be of service to control engineers primarily interested in applications, and to mathematicians with some exposure to control problems.

The major thrust in the development of observers for multivariable linear causal systems came from the introduction of state–space methods in the time-domain by Kalman in 1960. In the state–space approach, the dynamic behaviour of a system at any given instant is completely described in a finite-dimensional setting by the system state vector. The immediate impact of state–space methods was the strikingly direct resolution of many long-standing problems of control in a new multivariable system context; for example, pole-shifting compensation, deadbeat control, optimal linear regulator design and non-interacting control. These controllers are normally of the linear state feedback type and, if they are to be implemented, call for the complete availability of the state vector of the system. It is frequently the case, however, that even in low-order systems it is either impossible or inappropriate, from practical considerations, to measure all the elements of the system state vector. If one is to retain the many useful properties of linear state feedback control, it is necessary to overcome this problem of incomplete state information. The observer provides an elegant and practical solution to this problem. Now, an observer is an auxiliary dynamic system that reconstructs

the state vector of the original system on the basis of the inputs and outputs of the original system. The reconstructed state vector is then substituted for the inaccessible system state in the usual linear state feedback control law.

In keeping with the title "observers for linear systems", the framework is a finite-dimensional linear system one. Although the theory is mainly described in a linear state–space setting, frequent opportunity is taken to develop multivariable transfer-function methods and interpretations; so important if the designer is to fully exploit the structural properties of observers in a unified manner. Bearing in mind that an observer is itself a dynamic system and that it invariably constitutes the dynamic part of an otherwise static feedback control scheme, there is a marked interplay between observers, linear system theory and dynamic feedback compensation. This interaction is exploited in order to take full advantage of the latest and most significant advances in these subject areas. In particular, much use is made of a recurrent duality between state feedback control and state observation, and the fact that, for the most part, continuous-time and discrete-time problems are algebraically equivalent.

The text is organized as follows. Chapter 1 reviews the fundamental structural properties, namely observability and state reconstructability, that a system must possess for a corresponding state observer to exist. The basic theory of full-order observers, minimal-order observers and a special type of controller known as a dual-observer is introduced. In Chapter 2, the redundancy inherent in the structure of the minimal-order state observer is reduced by exhibiting the original system in various appropriate state–spaces. Chapter 3 examines the reconstruction of a linear function of the system state vector, typically a linear feedback control law, by an observer of further reduced dimension. In common with other chapters, the problem has two main aspects: the determination of the minimal order of the observer and stabilization of the observer. Chapter 4 explores further the possibilities of linear feedback control for systems with inaccessible state. Of particular interest is the construction of a dynamical controller based on the minimal-order state observer. The problem of observer design in order to reconstruct either the state vector or a linear state function of a discrete-time linear system in a minimum number of time steps is the subject of Chapter 5. Chapter 6 considers the problem of estimating the state of continuous-time and discrete-time linear stochastic systems in a least-square error sense, particularly where some but not all of the system measurements are noise-free. An important special case is when all the measurements contain additive white noise, in which case the optimal estimator is identical to the Kalman filter. In Chapter 7, adaptive observers and adaptive observer-based controllers are developed for continuous-time linear systems where *a priori* knowledge of the system parameters is lacking. The basic idea is that the observer estimates the unknown system parameters as well as the state variables of the system.

Chapter 8 undertakes a thorough examination of the complementary role multivariable frequency-response methods and state–space techniques have to play in observer-based system compensation. Using a complex-variable approach, some of the difficulties that may arise in the exclusive pursuit of time-domain methods of design, from the point of view of system robustness and controller instabilities, are highlighted. In Chapter 9, a polynomial-matrix approach is adopted for the synthesis of an observer-based compensator that further serves as a unifying link between transfer-function methods and state–space techniques. Chapter 10 establishes synthesis properties of state observers and linear function observers in terms of a few basic system concepts exhibited in a geometric state–space setting. The book closes in Chapter 11 with a brief discussion of extensions and applications.

Pains have been taken to make the text accessible to both engineers and mathematicians. Some acquaintance with linear algebra, the rudiments of linear dynamic systems and elementary probability theory is assumed. For ease of reference, however, a brief review of the more relevant background material is presented in two appendixes. Theorems, Propositions, etc. have been used to convey major results and summaries in a concise and self-contained fashion. The guiding idea is not rigour *per se*, but rather clarity of exposition. Proofs are usually given unless precluded by excessive length or complexity, in which case the appropriate reference is cited. It is intended that the notes and references which form an integral part of the text, should place the reader in a favourable position to explore the journal literature.

Liverpool
February 1983

J. O'Reilly

Acknowledgements

My debt to other investigators is obvious. Where possible I have endeavoured to discharge this debt by specific reference to other works, original papers, etc. This account of observers for linear systems is, in the last analysis, a personal one and will alas, inevitably overlook some contributions. To those who, unknown to me, have helped to shape the present perspective, I express my sincere gratitude if not by name.

Of those who have been most closely associated with my investigations, I feel especially grateful to my former supervisor, Dr M. M. Newmann, who introduced me to the fascinating study of observers. I also thank Professor A. P. Roberts, Dr G. W. Irwin and Dr J. W. Lynn for their encouragement and interest. Part of the manuscript was written during a short stay at the Coordinated Science Laboratory of the University of Illinois, for which I acknowledge the hospitality of Professor J. B. Cruz, Jr, Professor W. R. Perkins and Professor P. V. Kokotovic.

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Finally, I gratefully acknowledge the support of my family and friends in the face of the stresses radiating from me as the centre of this extra activity.

Contents

To my mother, my sister Ursula
and the memory of my father

Chapter 1

Elementary System and Observer Theory

1.1 INTRODUCTION

Since the re-emergence of state–space methods to form a direct multivariable approach to linear control system synthesis and design, a host of controllers now exist to meet various qualitative and quantitative criteria including system stability and optimality. A common feature of these control schemes is the assumption that the system state vector is available for feedback control purposes. The fact that complex multivariable systems rarely satisfy this assumption necessitates either a radical revision of the state–space method, at the loss of its most favourable properties, or the reconstruction of the missing state variables.

Adopting the latter approach, the state observation problem centres on the construction of an auxiliary dynamic system, known as a state reconstructor or observer, driven by the available system inputs and outputs. A block diagram of the open-loop system state reconstruction prcess is presented in Fig. 1.1. If, as is usually the case, the control strategy is of the linear state feedback type $u(t) = Fx(t)$, the observer can be regarded as forming part of a linear feedback compensation scheme used to generate the desired control approximation $F\hat{x}(t)$. This closed-loop observer-system configuration is depicted in Fig. 1.2.

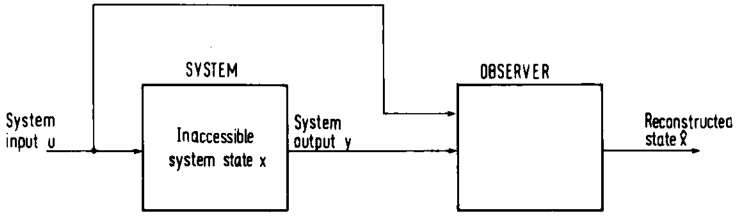

Fig. 1.1 Open-loop system state reconstruction.

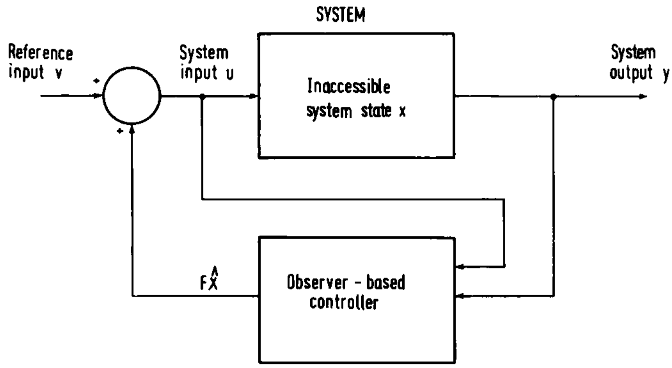

Fig. 1.2 Closed-loop observer-based control system.

The present chapter begins with an introduction to the state–space description of linear dynamical systems. Section 1.3 reviews the fundamental structural properties, namely controllability, reachability, observability and state reconstructability, that a linear system must possess for state feedback control and asymptotic state reconstruction by an observer. In Section 1.5 and Secton 1.6, the repective problems of linear state feedback control with accessible state vector and with inaccessible state vector are discussed. The resolution of the latter problem involves the asymptotic reconstruction of the inaccessible state variables by an observer of dynamic order equal to that of the original system. Section 1.7 sees a major simplification in the reduction of observer order by the number of available measurements of the system state variables to yield a state observer of minimal order. Fortunately, especially from implementation considerations, the parameters of most systems can reasonably be assumed to be constant. In this case, the appropriate minimal-order or full-order observer is time-invariant. Minimal-order observers for discrete-time linear systems are treated in Section 1.8. Finally, in Section 1.9 we reverse the fundamental process of one system observing another system to obtain a special type of controller known as a dual-observer.

1.2 LINEAR STATE–SPACE SYSTEMS

The dynamic behaviour of many systems at any time can be described by the continuous-time finite-dimensional linear system model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \tag{1.1}$$

$$y(t) = C(t)x(t)^* \tag{1.2}$$

* The more general output description $y = Cx + Hu$ is readily accommodated by redefining Equation (1.2) as $\bar{y} \triangleq y - Hu = Cx$.

where $x(t) \in R^n$ is the system state, $x(t_0) \in R^n$ is the state at the initial time t_0, $u(t) \in R^r$ is the control input, and the output $y(t) \in R^m$ represents those linear combinations of the state $x(t)$ available for measurement. The matrices $A(t)$, $B(t)$ and $C(t)$ are assumed to have compatible dimensions and to be continuous and bounded. Throughout the text, the term "linear system" is taken to mean a finite-dimensional linear dynamical system, it being understood that such a linear system is in fact an idealized (mathematical) model of an actual physical system. A solution of the vector differential equation (1.1) is given by the well-known variation of constants formula

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \lambda)B(\lambda)u(\lambda)\, d\lambda \tag{1.3}$$

where the *transition matrix* $\Phi(t, t_0)$ is the solution of the matrix differential equation

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I_n. \tag{1.4}$$

It is remarked that (1.3) holds for *all* t and t_0, and not merely for $t \geqslant t_0$. For the most part we shall deal with linear constant systems, otherwise known as linear *time-invariant* systems in which the defining matrices A, B and C are independent of time t.

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.5}$$

$$y(t) = Cx(t). \tag{1.6}$$

The transition matrix of (1.5) is given by

$$\Phi(t, t_0) = \exp A(t - t_0) \tag{1.7}$$

where the exponential function $\exp A(t - t_0)$ is defined by the absolutely convergent power series

$$\exp A(t - t_0) = \sum_{i=0}^{\infty} \frac{A^i(t - t_0)}{i!}. \tag{1.8}$$

1.2.1 Linearization

It is invariably the case that the true dynamical system, for which (1.1) and (1.2) represents a linear model, is in fact non-linear and is typically of the form

$$\dot{x}(t) = f[x(t), u(t), t] \tag{1.9}$$

$$y(t) = g[x(t), t]. \tag{1.10}$$

Nonetheless, a linear model of the form of (1.1) and (1.2) can be made to serve as an extremely useful approximation to the non-linear system (1.9) and (1.10)

by linearizing (1.9) and (1.10) about a nominal state trajectory $x_0(t)$ and a nominal input $u_0(t)$ where

$$\dot{x}_0(t) = f[x_0(t), u_0(t), t] \tag{1.11}$$

$$y_0(t) = y[x_0(t), t]. \tag{1.12}$$

That is, if one considers *small* perturbations $\delta x(t) \triangleq x(t) - x_0(t)$ and $\delta u \triangleq u(t) - u_0(t)$, one has from the Taylor's series expansions of (1.9) and (1.10) about $[x_0(t), u_0(t), t]$ that

$$\begin{aligned} f[x(t), u(t), t] = f[x_0(t), u_0(t), t] &+ A_0(t)\,\delta x(t) \\ &+ B_0(t)\,\delta u(t) + \alpha_0[x(t), u(t), t] \end{aligned} \tag{1.13}$$

$$g[x(t), t] = g[x_0(t), t] + C_0(t)\,\delta x(t) + \beta_0[\delta x(t), t] \tag{1.14}$$

where $\alpha_0[\delta x(t), \delta u(t), t]$ and $\beta_0[\delta x(t), t]$ denote second and higher-order terms in the Taylor series expansions. The matrices

$$A_0(t) \triangleq \left.\frac{\partial f}{\partial x}\right|_{[x_0, u_0, t]}, \quad B_0(t) \triangleq \left.\frac{\partial f}{\partial u}\right|_{[x_0, u_0, t]} \tag{1.15}$$

$$C_0(t) \triangleq \left.\frac{\partial g}{\partial x}\right|_{[x_0, u_0, t]} \tag{1.16}$$

are Jacobian matrices of appropriate dimensions, evaluated at the known system nominal values $[x_0(t), u_0(t), t]$. From (1.9) to (1.14), neglecting second and higher-order expansion terms, it is readily deduced that

$$\delta\dot{x}(t) = A_0(t)\,\delta x(t) + B_0\,\delta u(t) \tag{1.17}$$

$$\delta y(t) = C_0(t)\,\delta x(t). \tag{1.18}$$

The linearized perturbation model (1.17) and (1.18) is of the same linear form as (1.1) and (1.2), and yields a close approximation to the true non-linear system provided the higher-order expansion terms $\alpha_0[\delta x(t), \delta u(t), t]$ and $\beta_0[x(t), t]$ are *small* for all time t. We shall see presently how the validity of the linearized perturbation model, as characterized by the "smallness" of α_0 and β_0 is reinforced by the (linear) control objective of choosing $\delta u(t)$ so as to regulate $\delta x(t)$ to zero. Henceforth, our attention is focussed on the linear system model (1.1) and (1.2), bearing in mind that it may have arisen in the first place from the linearization of a non-linear process model just described.

1.2.2 Stability

A crucial question in systems theory is whether solutions of the non-linear system of differential equations (1.9) or the linear system of differential

equations (1.1) tend to increase indefinitely or not. Restricting our attention to linear time-varying bounded systems (1.1), the various different statements that one can make about the stability of the system coincide in the linear case.

Theorem 1.1 *Consider the linear system (1.1) where for positive constants* $c_1, \ldots, c_9$,

(*i*) $\|A(t)\| \leqslant c_1 < \infty$ *for all* t
(*ii*) $0 < c_2 \leqslant \|B(t)x\| \leqslant c_3 < \infty$ *for all* $\|x\| = 1$ *and all* t.

Then the following statements are equivalent:

(*a*) *Any uniformly bounded input*

$$\|u(t)\| \leqslant c_4 < \infty, \quad t \geqslant t_0 \tag{1.19}$$

gives rise to a uniformly bounded response for all $t \geqslant t_0$

$$\begin{aligned} \|x(t)\| &= \|\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\, d\tau\| \\ &\leqslant c_5(c_4, \|x_0\|) < \infty; \end{aligned} \tag{1.20}$$

(*b*) *The equilibrium state* $x_e(t) = 0$ *of the input-free system is* uniformly asymptotically stable*;*
(*c*) *There exist positive constants* c_6 *and* c_7 *such that*

$$\|\Phi(t, t_0)\| \leqslant c_6 \exp[-c_7(t - t_0)] \quad \textit{for all } t \geqslant t_0 \tag{1.21}$$

(*d*) *Given any positive-definite matrix* $Q(t)$ *continuous in* t *and satisfying for all* $t \geqslant t_0$

$$0 \leqslant c_8 I \leqslant Q(t) \leqslant c_9 I < \infty \tag{1.22}$$

there exists a scalar (Lyapunov) function

$$V(x(t), t) = x'(t)P(t)x(t) \tag{1.23}$$

such that its derivative along the free motion of (1.1) is

$$\dot{V}(x(t), t) = -x'(t)Q(t)x(t) \tag{1.24}$$

Proof See [K15]. □

Proposition (c) of Theorem 1.1 is frequently taken as a definition of *uniform exponential stability* of the input-free system (1.1). Proposition (d) concerns stability in the sense of the so called "second method" of Lyapunov. Physically, the Lyapunov function $V(x)$ can be thought of as a generalized energy function of an isolated system whereby if $dV(x)/dt$ is negative, $V(x)$ decays to its minimum value $V(x_e)$ at the equilibrium state x_e.

Corollary 1.1 *The linear time-invariant system (1.5) is asymptotically stable (exponentially stable, stable in the sense of Lyapunov) if and only if all the eigenvalues of A have negative real parts.*

Analogous stability results hold for discrete-time linear systems of the form of (1.27) and are to be found in Kalman and Bertram [K16].

1.3 CONTROLLABILITY AND OBSERVABILITY

1.3.1 Reachability and controllability

The prime objective of linear control theory is the regulation of the state $x(t)$ to some desired reference state by approriate manipulation of the control input vector $u(t)$. Depending on whether one wishes to transfer the zero state to some desired trajectory or transfer any arbitrary initial state to the zero state, the ability to exert the required control action is a structural characteristic of the system (1.1) known as reachability or controllability.

Definition 1.1 The state of the continuous-time linear dynamical system (1.1) is said to be *reachable (from the zero state)* at time t if there exists a $\tau \leqslant t$ and an input $u \in R^r$ which transfers the zero state at time τ to the state x at time t.

Definition 1.2 The continuous-time linear dynamical system (1.1) is said to be *controllable (to the zero state)* if, given any initial state $x(\tau)$, there exists a $t \geqslant \tau$ and a $u \in R^r$ such that $x(t) = 0$.

Theorem 1.2 *A necessary and sufficient condition for system reachability [controllability] at time τ is that $W(\tau, t)$ [$W(s, \tau)$] is positive definite for some $t \geqslant \tau$ [$s \leqslant \tau$] where $W(.,.)$ is an $n \times n$ Gramian matrix defined by*

$$W(\tau, t) = \int_\tau^t \Phi(\tau, \lambda)B(\lambda)B'(\lambda)\Phi'(\tau, \lambda)\, d\lambda$$

Proof See [K18] or [W5]. □

A stronger type of reachability [controllability] known as *uniform* reachability [*uniform* controllability] holds if and only if for some $\sigma > 0$ there exists a positive constant α_1 such that $W(\tau, \tau + \sigma) \geqslant \alpha_1$ for all τ. Roughly speaking, the property of uniform reachability or uniform controllability refers to the fact that the property is independent of the initial time τ.

For linear time-invariant systems we have a simple algebraic criterion for (complete) controllability.

Theorem 1.3 *The linear time-invariant system (1.5) or the pair (A, B) is completely controllable if and only if*

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n \tag{1.25}$$

Proof See [K18]. □

The fact [K18] that for time-invariant linear systems

$$\dim \operatorname{range} W(\tau, t) = \operatorname{rank}[B, AB, \ldots, A^{n-1}B] \tag{1.26}$$

for $t > \tau$ implies that reachability and controllability in Theorem 1.2 are equivalent. In other words, the linear time-invariant system (1.5) is completely reachable if and only if it is completely controllable as in Theorem 1.3.

Analogous definitions and conditions for reachability and controllability hold for discrete-time linear systems described by the finite-dimensional difference equation model

$$x_{k+1} = A_k x_k + B_k u_k, \quad k = 0, 1, 2, \ldots \tag{1.27}$$

$$y_k = C_k x_k \tag{1.28}$$

1.3.2 Observability and reconstructibility

A primary objective, and the major theme of this book is the reconstruction of the system state from the output measurements. The potential for state reconstruction is a structural characteristic of the linear system (1.1), (1.2) and can be defined in two different ways depending on whether one wishes to deduce the present state from past or future output measurements. We first characterize the cases where this cannot be done.

Definition 1.3 Let $y(t; \tau, x, u)$ denote the output response of the linear system (1.1), (1.2) to the initial state $x(\tau)$. Then the (present) state $x(\tau)$ of the linear system (1.1), (1.2) is *unobservable* if the (future) output

$$y(t; \tau, x, 0) = 0$$

for all $t \geqslant \tau$.

Definition 1.4 The (present) state $x(\tau)$ of the linear system (1.1), (1.2) is *unreconstructible* if the (past) output

$$y(\sigma; \tau, x, 0) = 0$$

for all $\sigma \leqslant \tau$.

Since normally only past system measurements are available, it is rather more important to characterize the ability of reconstructing the state from past

measurements in Definition 1.4 than from future measurements in Definition 1.3.

Theorem 1.4 *The state $x(\tau)$ of the linear system (1.1), (1.2) is unreconstructible if and only if $x \in$ kernal $M(s, \tau)$ for all $s \leqslant \tau$, where*

$$M(s, \tau) = \int_s^\tau \Phi'(\sigma, \tau)C'(\sigma)C(\sigma)\Phi(\sigma, \tau)\, d\sigma \tag{1.29}$$

Proof (*necessity*) The formula

$$x'M(s, \tau)x = \int_s^\tau \|C(\sigma)\Phi(\sigma, \tau)x\|^2\, d\sigma$$

shows that if $x(\tau)$ is unreconstructible, then $x'M(s, \tau)x = 0$ for all s. Since M is symmetric and non-negative definite, $M = N'N$. Then $x'Mx = \|Nx\|^2 = 0$ which implies that $Nx = 0$ and so $Mx = N'Nx = 0$. □

Proof (*sufficiency*) If $x \in$ kernal M, then $x'Mx = 0$, and the same formula shows that $C(\sigma)\Phi(\sigma, \tau)x = 0$ for all $\sigma \leqslant \tau$. Q.E.D. □

A complementary characterization of unobservable systems is as follows.

Theorem 1.5 *The state $x(\tau)$ of the linear system (1.1), (1.2) is unobservable if and only if $x \in$ kernal $\hat{M}(\tau, t)$ for all $t \geqslant \tau$, where*

$$\hat{M}(\tau, t) = \int_\tau^t \Phi'(\sigma, \tau)C'(\sigma)C(\sigma)\Phi(\sigma, \tau)\, d\sigma \tag{1.30}$$

The following corollary to Theorem 1.4 and Theorem 1.5 is immediate.

Corollary 1.2 *The linear system (1.1), (1.2) or the pair $(A(t), C(t))$ is completely observable [completely state reconstructible] if and only if $\hat{M}(\tau, t)$ $[M(s, \tau)]$ is positive definite for $t \geqslant \tau$ $[s \leqslant \tau]$ where the Gramian matrices $\hat{M}(\tau, t)$ and $M(s, \tau)$ are defined by (1.30) and (1.29).*

A stronger type of observability and state reconstructibility is obtained by imposing further conditions on the Gramian matrices $\hat{M}$ and M.

Theorem 1.6 *The linear system (1.1), (1.2) or the pair $(A(t), C(t))$ is* uniformly *completely observable* [uniformly *completely state reconstructible*] *if and only if for some $\sigma > 0$ there exist positive constants α_i, $i = 1, 2$ $[\alpha_i, i = 3, 4]$ such that*

$$0 < \alpha_1 I \leqslant \hat{M}(\tau, \tau + \sigma) \leqslant \alpha_2 I \tag{1.31}$$

$$[0 < \alpha_3 I \leqslant M(\tau - \sigma, \tau) \leqslant \sigma_4 I] \tag{1.32}$$

holds for all τ.

In the time-invariant case, state reconstructibility is equivalent to observability.

Theorem 1.7 *The linear time-invariant system (1.5), (1.6) or the pair (A, C) is completely observable if and only if it is completely state reconstructible.*

For linear time-invariant systems we have a simple algebraic criterion for complete observability or complete state reconstructibility.

Theorem 1.8 *The linear time-invariant system (1.5), (1.6) or the pair (A, C) is completely observable [completely state reconstructible] if and only if*

$$\operatorname{rank} P = \operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{1.33}$$

Proof (sufficiency) **By constancy, we may assume without loss of generality that $\sigma = 0$. If the system state is unreconstructible, then by Definition 1.4 there** is a $x \neq 0$ such that

$$y(0, \tau, x, 0) = C\Phi(0, \tau)x = 0 \quad \text{for all } \tau \geqslant 0.$$

Repeated differentiation of y with respect to τ and then letting $\tau = 0$ gives

$$Cx = 0, CAx = 0, \ldots$$

So $x \neq 0$ is orthogonal to every element of the *observability matrix*

$$P = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and we have a contradiction if the rank of P is n.

Proof (necessity) Suppose rank $P < n$. Then there is an $x \neq 0$ in R^n such that

$$Cx = 0, CAx = 0, \ldots, CA^{n-1}x = 0.$$

By the Cayley-Hamilton theorem (Appendix A), we then also have $CA^n x = 0$, and it follows by induction that $CA^i x = 0$ for all $i \geqslant 0$. Thus

$$\left[C - CA\tau + \frac{CA^2\tau^2}{2!}, \ldots\right]x = 0$$

or, by (1.7) and (1.8),

$$C\Phi(0, \tau)x = y(0, \tau, x, 0) = 0.$$

In conclusion, if rank $P < n$, we have by Definition 1.4 that there is an $x \neq 0$ which is not reconstructible. Q.E.D. □

The fact that the observability matrix P in (1.33) is of rank n means that it contains n linearly independent rows among the nm rows forming P. It is possible that n such linearly independent rows can be found among the qm rows forming the *partial observability matrix.*

$$P_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}, \quad 1 \leqslant q \leqslant n. \tag{1.34}$$

The least integer q, called ν, such that rank $P_q = n$ is known as the *observability index* of the system pair (A, C).

Similarly, the least integer q, called μ, such that

$$\operatorname{rank}\,[B, AB, \ldots, A^{q-1}B] = n, \quad 1 \leqslant q \leqslant n \tag{1.35}$$

is known as the *controllability index* of the system pair (A, B).

An extension of the observability criterion (1.33) to linear time-varying systems may be made in defining a type of uniform differential observability. That is, the system (1.1) and (1.2) or the pair $(A(t), C(t))$ is said to be uniformly (differentially) observable if and only if rank $M(t) = n$ for all $t \geqslant t_0$ where the $nm \times n$ observability matrix

$$M(t) \triangleq \begin{bmatrix} Q_1(t) \\ Q_2(t) \\ \vdots \\ Q_n(t) \end{bmatrix} \tag{1.36}$$

and

$$Q_1(t) = C(t)$$

$$Q_i(t) = Q_{i-1}(t)A(t) + \dot{Q}_{i-1}, \quad i = 2, 3, \ldots, n.$$

Analogous definitions and conditions are made for the discrete-time linear system (1.27) and (1.28) where the control input u_k is assumed to be known.

Definition 1.5 The discrete-time linear system (1.27), (1.28) or the pair (A_k, C_k) is said to be completely ν-step observable [completely ν-step state reconstructible] at time μ if knowledge of $y(\mu), y(\mu+1), \ldots, y(\mu+\nu-1)$ [$y(\mu-\nu+1), \ldots, y(\mu)$] is sufficient to determine $x(\mu)$.

Theorem 1.9 *The discrete-time linear system (1.27), (1.28) is completely ν-step observable [completely ν-step state reconstructible] at time μ if and only if [if]*

$$\operatorname{rank}\,[C'_\mu, \Omega'(\mu,\mu)C'_{(\mu+1)}, \ldots, \Omega'(\mu+\nu-2,\mu)C'_{(\mu+\nu-1)}] = n \tag{1.37}$$

where

$$\Omega(k,j) \triangleq A_k A_{k-1} \cdots A_{j+1} A_j, \quad k \geqslant j.$$

Proof See [W4]. □

Note that the criterion (1.37) is only sufficient for complete ν-step state reconstructibility at μ unless the system matrix $A(.)$ is invertible over $[\mu, \mu+\nu-1]$. For time-invariant systems, (1.37) reduces to the well known condition

$$\operatorname{rank}\,[C', A'C', \ldots, A'^{\nu-1}C'] = n \tag{1.38}$$

for some integer $\nu \leqslant n$.

1.3.3 Duality of state reconstruction and control

A comparison of the definitions and results on observability (state reconstructibility) with those on reachability (controllability) reveals a striking symmetry between these properties. This symmetry between state reconstruction and control is enshrined in the *Principle of Duality* [K6, K18].

Definition 1.6 The linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.39}$$

$$y(t) = C(t)x(t) \tag{1.40}$$

is the *dual* of the system (and *vice versa*)

$$\dot{x}^*(t) = A_1(t)x^*(t) + B_1(t)u^*(t) \tag{1.41}$$

$$y^*(t) = C_1(t)x^*(t) \tag{1.42}$$

where for fixed τ and arbitrary t

$$A_1(\tau+t) = A'(\tau-t)$$

$$B_1(\tau+t) = C'(\tau-t)$$

$$C_1(\tau+t) = B'(\tau-t)$$

$$\Phi(\tau, \tau+t) = \Phi'(\tau-t, \tau).$$

For time-invariant systems, the duality is exhibited by the relations

$$A_1 = A', \quad B_1 = C', \quad C_1 = B'. \tag{1.43}$$

The Principle of Duality is a useful one in that it allows state reconstruction problems to be reformulated as dual control problems, and *vice versa*. In particular, inspection of the preceding results immediately gives rise to the following theorem.

Theorem 1.10 *The system (1.39), (1.40) is (uniformly) completely reachable [(uniformly) completely controllable] if and only if the dual system (1.41), (1.42) is (uniformly) completely observable [(uniformly) completely state reconstructible], and vice versa.*

An analogous duality holds for the discrete-time system (1.27) and (1.28) whereby the system (1.27), (1.28) is completely ν-step reachable [completely ν-step controllable] at time μ if and only if the system

$$x^*(k-1) = A'(k)x^*(k) + C'(k)u^*(k) \tag{1.44}$$

$$y^*(k) = B'(k)x^*(k) \tag{1.45}$$

with the time scale reversed about μ, is completely ν-step observable [completely ν-step state reconstructible] at time μ.

Moreover, taking all these results together, one has the overall description of any linear dynamical system in the *canonical structure theorem.*

Theorem 1.11 *Any finite-dimensional linear system may be decomposed into four subsystems: (1) a controllable and observable subsystem; (2) a subsystem which is controllable but not observable; (3) a subsystem which is observable but not controllable; and finally, (4) a subsystem which is neither controllable nor observable.*

Despite its simplicity, Theorem 1.11 has profound implications: only in case (1) can effective control be exerted on the system state and can the state be observed from the available measurements. Fortunately, most linear systems are both completely controllable and observable. In any case, for time-invariant systems (1.5), (1.6) the assumption can be checked by the conditions of Theorem 1.3 and Theorem 1.8 or, equivalently, by the (numerically) simpler conditions of Theorem 1.12.

Theorem 1.12 *A necessary and sufficient condition for the system (1.5) and (1.6) to be completely controllable [completely observable] is that*

$$\operatorname{rank}\,[A - \lambda_i I, B] = n \tag{1.46}$$

$$\left[\operatorname{rank}\begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} = n\right] \tag{1.47}$$

where λ_i is the ith, $i = 1, \ldots, n$, eigenvalue of A.

Proof See [R10]. □

In the context of the theory of matrix polynomials, about which we shall say more in Chapter 9, the matrices $\lambda I - A$ and B satisfying condition (1.46) are said to be *relatively left prime* or *left coprime*. Similarly, the matrices $\lambda I - A$ and C satisfying condition (1.47) are said to be *relatively right prime* or *right coprime*.

The conditions for controllability and observability can be weakened to those of *stabilizability* and *detectability*.

Definition 1.7 The linear time-invariant system (1.5), (1.6) is stabilizable [detectable] if and only if the uncontrollable [unobservable] states are stable. An immediate consequence of this definition is the following result.

Theorem 1.13 *Any asymptotically stable time-invariant system (1.5), (1.6) is stabilizable and detectable. Any completely controllable [completely observable] system is stabilizable [detectable].*

1.4 TRANSFER-FUNCTION REPRESENTATION OF TIME-INVARIANT SYSTEMS

The finite-dimensional linear state–space model (1.5) and (1.6) is an *internal* dynamical representation of a system in the time-domain. It is often useful to analyse the *external* input–output properties of linear time-invariant systems by extending the well-known notion of a scalar transfer function to multivariable state-space systems. We define the Laplace transform of a time-dependent vector $z(t)$ as follows

$$z(s) = \mathscr{L}[z(t)] = \int_0^\infty z(t)\, e^{-st}\, dt \tag{1.49}$$

where s is a complex variable.

Taking Laplace transforms of the state representation (1.5) and (1.6) yields

$$x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s) \tag{1.50}$$

$$y(s) = Cx(s). \tag{1.51}$$

The matrix function $(sI - A)^{-1}$ is known as the *resolvent* of A. A comparison of (1.50) with the time-domain expression (1.3) and (1.7) reveals that the resolvent matrix is given by

$$(sI - A)^{-1} = \mathscr{L}[\exp At] = \mathscr{L}[e^{At}] \tag{1.52}$$

Disregarding the information about the internal initial state $x(0)$, a transfer-function matrix $T(s)$ relating the external system input transform $u(s)$ to the external system output transform $y(s)$ of (1.50) and (1.51) is given by

$$T(s) = C(sI - A)^{-1}B \tag{1.53}$$

or

$$T(s) = C\frac{\operatorname{adj}(sI - A)B}{\det(sI - A)} \tag{1.54}$$

where $\operatorname{adj}(sI - A)$ is the adjoint of the polynomial matrix $sI - A$. Each element $T_{ij}(s)$ of the $m \times r$ transfer-function matrix $T(s)$ is the transfer function from the jth component of the input ($j = 1, \ldots, r$) to the ith component of the output ($i = 1, \ldots, m$). For linear time-invariant systems (1.5), (1.6) that are both completely controllable and completely observable, the n eigenvalues of A, or the roots of the characteristic polynomial $\det(sI - A)$, are identical to the *poles* of the open-loop transfer-function matrix $T(s)$. In view of the fact that $\operatorname{adj}(sI - A)$ is an $n \times n$ polynomial matrix whose entries are polynomials in the complex variable s of degree no greater than $n - 1$, it is clear that the numerator degree of each transfer-function entry $T_{ij}(s)$ of $T(s)$ in (1.54) will be strictly less than the denominator degree: such a $T(s)$ is known as a *strictly proper* transfer-function matrix [W10].

The fact that the transfer-function matrix $T(s)$ describes only the external (input–output) behaviour of a system means that it is independent of any particular choice of the internal state. Specifically, two systems $\{A, B, C\}$ and $\{\bar{A}, \bar{B}, \bar{C}\}$ related by the equivalence transformation $\bar{x} = Qx$, where Q is any $n \times n$ non-singular matrix, have the *same* transfer-function matrix

$$\begin{aligned} T(s) &= C(sI - A)^{-1}B = CQ^{-1}Q(sI - A)^{-1}Q^{-1}QB \\ &= \bar{C}(sI - \bar{A})^{-1}\bar{B} \end{aligned} \tag{1.55}$$

although their internal state vectors x and $\bar{x}$ are, in general, quite *different*.

Conversely, given the transfer-function matrix of a system, it is of interest to determine a state–space representation in the time-domain.

Definition 1.8 A *realization* of a proper rational transfer-function matrix $T(s)$ is any state representation whose transfer-function matrix is $T(s)$, as in (1.55).

A state–space realization $\{A, B, C\}$ of the transfer–function matrix $T(s)$ is

minimal if it has the lowest order or state dimension among all realizations having the same transfer–function matrix. Moreover, in accordance with (1.55) any two minimal realizations $\{A, B, C\}$ and $\{\bar{A}, \bar{B}, \bar{C}\}$ are equivalent; that is, $\{\bar{A}, \bar{B}, \bar{C}\} = \{QAQ^{-1}, QB, CQ^{-1}\}$ for some non-singular matrix Q.

Theorem 1.14 *An n-dimensional realization $\{A, B, C\}$ of $T(s)$ is minimal if and only if it is both completely controllable and completely observable.*

Proof See [W10] for a full proof. Necessity follows from the fact that, in accordance with Theorem 1.11, uncontrollable and unobservable system modes can be deleted leaving a realization of lower dimension which is both completely controllable and completely observable; hence the necessity of controllability and observability for minimality. □

1.5 LINEAR STATE FEEDBACK

Before embarking on a discussion of the role of observers in reconstructing the state vector of linear systems with inaccessible state in Section 1.6, it is instructive to first consider the use of linear state feedback for systems with completely accessible state. In Section 1.3, we characterized the potential for transferring any arbitrary initial state of a time-invariant linear system to a desired reference (null) state as a property of the system matrices A and B known as controllability (reachability). It is now considered how this control action may be effected by linear state feedback. Suppose that the state $x(t)$ of the open-loop system (1.5) is completely accessible; then a linear feedback control law of the type

$$u(t) = Fx(t) + v(t) \tag{1.56}$$

where $v(t)$ is an external reference input (sometimes zero) to the system, when applied to (1.5), results in the closed-loop system described by

$$\dot{x}(t) = (A + BF)x(t) + Bv(t). \tag{1.57}$$

The state vector of (1.57) may be asymptotically driven to the desired null vector if a feedback gain matrix F can be chosen to assign the eigenvalues of the matrix $A + BF$ to the left-hand side of the complex plane in accordance with Corollary 1.1. The ability to achieve arbitrary eigenvalue assignment through some choice of F is established in a fundamental theorem, generally attributed to Wonham [W13], where it is assumed that complex eigenvalues only occur in conjugate pairs.

Theorem 1.15 *Corresponding to the real matrices A and B, there is a real matrix F such that the set of eigenvalues of $A + BF$ can be arbitrarily assigned if and only if the pair (A, B) is completely controllable.*

Proof A proof of this theorem is given by Wonham [W13], Willems and Mitter [W5] and Wolovich [W10]. The design procedures of Section 2.5 constitute constructive proofs of the dual result of Theorem 1.16 for single-output and multi-output systems respectively. □

Thus, complete availability of the system state vector and the consequent application of the linear feedback control law (1.56) to the linear controllable system (1.5) allows the dynamic response of the closed-loop system to be asymptotically adjusted to zero through arbitrary (stable) assignment of the eigenvalues of $A + BF$ to the left-hand side of the complex plane. In view of Definition 1.7 we have the following corollary.

Corollary 1.3 *The linear time-invariant system (1.5) is stabilizable if and only if there exists a real matrix F such that $A + BF$ is a stability matrix, i.e., has all its eigenvalues in the left-hand side of the complex plane.*

A discussion of suitable methods of choosing the feedback gain matrix so as to improve the closed-loop system response is the subject of Section 4.2.

1.6 STATE RECONSTRUCTION AND THE INACCESSIBLE STATE FEEDBACK CONTROL PROBLEM

We have just seen that one can alter the dynamic response of a controllable system at will through the application of a linear state feedback controller (1.56). Such an application presumes that the structure of the system is known *a priori*, and that the current state $x(t)$ of the system is completely available. Leaving aside the determination of unknown parameters of the system until Chapter 7, we consider the common situation where the state is not completely available but the system parameters are known. That is, we consider the continuous-time linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.58}$$

$$y(t) = Cx(t) \tag{1.59}$$

where only m linear combinations of the internal state vector $x(t) \in R^n$ are availablein the output vector $y(t) \in R^m$, $m < n$. The systems matrices A, B and C are assumed to be known. In order to implement the linear feedback control law (1.56), or out of independent interest, a means of reconstructing the

unavailable state $x(\cdot)$ is sought from a knowledge of the actual inputs $u(\cdot)$ and outputs $y(\cdot)$ of the system (1.58) and (1.59).

In Section 1.3, we characterized the potential for reconstruction of the system state from a knowledge of future or past output measurements in terms of conditions on the system matrices A and C known respectively as observability and state reconstructability. Consideration is now given to the means of realizing this potential through the design of a dynamic observer which operates on past system measurements where we restrict our attention to time-invariant systems of the type (1.58) and (1.59).

Definition 1.8 The state reconstruction problem is one of determining the state $x(\tau)$ of the system (1.58) at time τ from a knowledge of the output measurements $y(\sigma)$, $\sigma \leqslant \tau$.

An auxiliary dynamical system which will somehow reconstruct the state $x(\tau)$ from $y(\sigma)$, $\sigma \leqslant \tau$, is described by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G(y(t) - C\hat{x}(t)) \tag{1.60}$$

where $\hat{x}(t) \in R^n$ denotes the reconstructed state at any time $t \geqslant t_0$. This auxiliary system is known as a *full-order observer* or an *identity observer*, and is coupled to the original system through the available system inputs and outputs. The first two terms on the right-hand side of (1.60) model the dynamics of the system (1.58), while the last term, based on the measurableoutputs, reflects the mismatch in the modelling process. Although in this sense the observer may be regarded as a model of the system, it is of engineering significance that, unlike the original system which consists of physical components, an observer can be realized quite cheaply by an analogue or digital simulation.

Representing the mismatch in the modelling process by the state reconstruction error

$$e(t) \triangleq \hat{x}(t) - x(t) \tag{1.61}$$

we have from (1.58) to (1.61) that

$$\dot{e}(t) = (A - GC)\, e(t), \quad t \geqslant t_0. \tag{1.62}$$

The remarkable fact that the error system (1.62) is the mathematical dual (see Definition 1.6) of the closed-loop system (1.57) means that, as a dual to Theorem 1.15, one has the corresponding fundamental result.

Theorem 1.16 *Corresponding to the real matrices A and C, there is a real matrix G such that the set of eigenvalues of $A - GC$ can be arbitrarily assigned*

(subject to complex eigenvalues occurring in conjugate pairs) if and only if the pair (A, C) is completely observable.

In the light of Definition 1.7 we have the following corollary.

Corollary 1.4 *The linear time-invariant system (1.58) and (1.59) is detectable if and only if there exists a real matrix G such that $A - GC$ is a stability matrix, i.e., has all its eigenvalues in the left-hand side of the complex plane.*

It is of interest to note that in the unlikely event that the initial system state $x(t_0)$ is known, one could set $\hat{x}(t_0) = x(t_0)$ which by (1.61) and (1.62), would yield $\hat{x}(t) \equiv x(t)$ for all $t \geqslant t_0$, and any gain G and input $u(t)$: that is, *perfect* state reconstruction would be achieved. In general, however, $x(t_0)$ is unknown. The significance of Theorem 1.16 then is that, regardless of its initial value $e(t_0)$, the state reconstruction error $e(t)$ may be asymptotically reduced to zero if a gain G is chosen to assign the eigenvalues of the matrix $A - GC$ to the left-hand side of the complex plane. The actual computation of such an error stabilizing gain is a non-trivial exercise and is deferred to Chapter 2. Other methods of choosing the observer gain matrix in a minimum mean-integral-squared error sense for time-invariant systems and in a linear least-squares sense for stochastic systems are treated in Chapters 4 and 6 respectively.

Having obtained an observer (1.60) which can be made to asymptotically reconstruct the state of the system (1.58), (1.59), it is natural to enquire as to the effect of substituting this state estimate in the linear feedback control law (1.56) to give

$$u(t) = F\hat{x}(t) + v(t). \tag{1.63}$$

Obviously in the steady state, there will be no degradation of performance since the error $e(t)$ in the state estimate will be zero. In order to investigate the initial transient performance when $e(t) \neq 0$, we consider the closed-loop observer-based control system, represented in Fig. 1.2, where the state estimate $\hat{x}(t)$ in (1.63) is generated by the full-order observer (1.60) for all $t \geqslant t_0$. Of particular interest is whether or not the observer will impair the stability of the closed-loop system.

The closed-loop system that results from the interconnection of the *dynamic* observer-based controller (1.63) and (1.60) to the open-loop system (1.58), (1.59) is described by the composite system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BF \\ GC & A - GC + BF \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v(t). \tag{1.64}$$

By definition of the reconstruction error (1.61), the system (1.64) may be transformed to the composite system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t). \tag{1.65}$$

The transformation between $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ and $\begin{bmatrix} x \\ e \end{bmatrix}$ is linear and non-singular so that the characteristic polynomial of the composite system (1.64) is quite the same as that of (1.65). But the characteristic polynomial

$$\det\left(\lambda I_{2n} - \begin{bmatrix} A + BF & BF \\ 0 & A - GC \end{bmatrix}\right)$$
$$= \det(\lambda I_n - A - BF) \times \det(\lambda I_n - A + GC). \tag{1.66}$$

Thus, the set of $2n$ eigenvalues of the overall closed-loop system (1.64) is made up of the eigenvalues of the closed-loop system matrix (1.57) assuming complete knowledge of the states and the eigenvalues of the observer matrix $A - KC$. Combining Theorem 1.15 and Theorem 1.16, we now have the striking result on the eigenvalue assignability and asymptotic stability of the overall closed-loop control system (1.64).

Theorem 1.17 *If the linear system (1.58) and (1.59) is completely controllable and completely observable, there exist gain matrices F and G such that the $2n$ eigenvalues of the system matrix of the closed-loop system (1.64) can be arbitrarily assigned, in particular to positions in the left-half complex plane.*

On the other hand, where one wishes to exert a control or stabilizing action only over those system modes which are open-loop unstable, one has by Corollary 1.3 and Corollary 1.4 the corresponding useful result.

Corollary 1.5 *If the linear system (1.58) and (1.59) is stabilizable and detectable, there exist gain matrices F and G such that the system matrix of the closed-loop system (1.64) is a stability matrix, i.e., has all its $2n$ eigenvalues on the left-hand side of the complex plane.*

Theorem 1.17 and Corollary 1.5 testify to a *separation principle* in observer-based controller design; namely the full-order observer (1.60) and the linear feedback controller (1.63) may be designed separately by the independent choice of the gain matrices F and G. We shall have frequent occasion to use variants of this fundamental and propitious result in succeeding chapters whenever we met with linear feedback control problems with inaccessible state. It is left as a simple exercise to show that in the time-varying case, asymptotic stability of the closed-loop system is ensured by that of $A(t) + B(t)F(t)$ and $A(t) - G(t)C(t)$. Finally, it is worth noting that, compared to

the accessible state case, the transient response of the closed-loop system (1.65) will, however, be impaired by non-zero values of reconstruction error $e(t)$. The importance of an observer as an asymptotic state reconstructor in the feedback control loop (Fig. 1.2) is that this degradation diminishes exponentially with the exponential decay of $e(t)$.

1.7 MINIMAL-ORDER OBSERVERS FOR LINEAR CONTINUOUS-TIME SYSTEMS

The full-order observer (1.60) is an auxiliary system of order n, equal to the state dimension of the original system (1.1). Notwithstanding the simplicity of its construction, it also possesses a measure of redundancy. Recalling that our objective is to reconstruct the system state $x(t) \in R^n$, the presence of m linear combinations of the state in the outputs (1.2) suggests that the remaining $n - m$ linear combinations may be reconstructed by an observer of order no greater than $n - m$. Such an observer is called a *minimal-order observer* or *reduced-order observer* in that the dimension of the observer state vector is $n - m$ as opposed to n for a full-order observer.

Thus, given m linear state combinations in the outputs (1.2) we wish to generate the remaining $n - m$ linear state combinations

$$z(t) = T(t)x(t), \quad z(t) \in R^{n-m}. \tag{1.67}$$

Combining (1.2) and (1.67),

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} C(t) \\ T(t) \end{bmatrix} x(t) \tag{1.68}$$

so that the complete state vector is determined exactly through

$$x(t) = \begin{bmatrix} C(t) \\ T(t) \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \tag{1.69}$$

$$= [V(t), P(t)] \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \tag{1.70}$$

where $T(t)$ is so chosen that the required matrix inverse exists. Thus, all would be well with our state reconstruction procedure if $z(t)$ in (1.67) did indeed represent $n - m$ linear combinations of the state $x(t)$. Since, however, $T(t)x(t)$, unlike $y(t)$, cannot be measured exactly, it must be reconstructed using an auxiliary dynamical system of order $n - m$

$$\dot{z}(t) = D(t)z(t) + E(t)y(t) + G(t)u(t) \tag{1.71}$$

known as a minimal-order state observer. Similar to (1.69), the reconstructed

state is given by

$$\hat{x}(t) = P(t)z(t) + V(t)y(t) \tag{1.72}$$

while the acceptability of $z(t)$ as an approximation for $T(t)x(t)$ is characterized by the observer reconstruction error

$$\varepsilon(t) \triangleq z(t) - T(t)x(t). \tag{1.73}$$

Using (1.1), (1.2), (1.71) and (1.73), the dynamics of this observer error are

$$\begin{aligned}\dot{\varepsilon}(t) = D(t)\varepsilon(t) &+ (D(t)T(t) - T(t)A(t) + E(t)C(t) - \dot{T}(t))x(t)\\ &+ (G(t) - T(t)B(t))u(t).\end{aligned} \tag{1.74}$$

Equation (1.74) reduces to the homogeneous equation

$$\dot{\varepsilon}(t) = D(t)\varepsilon(t) \tag{1.75}$$

provided the observer matrices $T(t)D(t)$, $E(t)$ and $G(t)$ satisfy

$$D(t)T(t) - T(t)A(t) = \dot{T}(t) - E(t)C(t) \tag{1.76}$$

and

$$G(t) = T(t)B(t). \tag{1.77}$$

Also, from (1.68) and (1.70), we have that

$$\begin{bmatrix} C(t) \\ T(t) \end{bmatrix}^{-1} = [V(t), P(t)]. \tag{1.78}$$

Equations (1.76) to (1.78) are known as the Luenberger observer constraint equations which the parameter matrices $T(t)$, $D(t)$, $E(t)$, $G(t)$, $P(t)$ and $V(t)$ of the minimal-order observer (1.71) and (1.72) are required to satisfy. Furthermore, denoting the actual state reconstruction error by

$$e(t) \triangleq \hat{x}(t) - x(t) \tag{1.79}$$

we have by (1.2), (1.72) and (1.78) that

$$e(t) = P(t)\varepsilon(t) + (P(t)T(t) + V(t)C(t) - I_n)x(t) \tag{1.80}$$

which by invoking (1.78) reduces to

$$e(t) = P(t)\varepsilon(t). \tag{1.81}$$

Thus the accuracy of the state reconstruction is a linear function of that of the observer state vector $z(t)$. If the initial system state $x(t_0)$ is known and we set $z(t_0) = T(t_0)x(t_0)$, Equation (1.75) implies that $\varepsilon(t) \equiv 0$ for all $t \geqslant t_0$ and by (1.81) perfect state reconstruction ensues. In general, however, $x(t_0)$ is unknown and the best that can be done is to make a guess for $z(t_0)$ in the range space of $T(t_0)$; i.e. set $z(t_0) = T(t_0)x_g$ for some $x_g \in R^n$. Fortunately, however, if

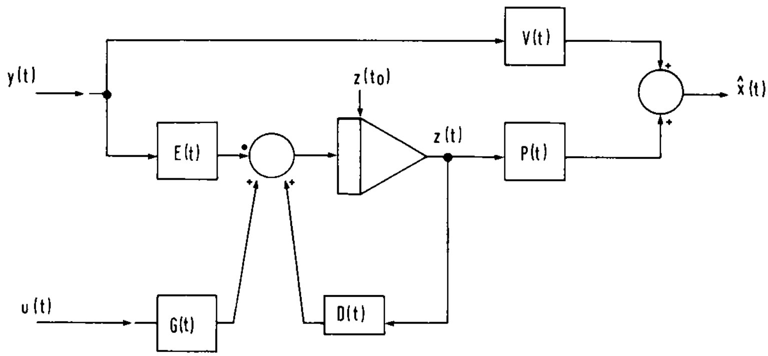

Fig. 1.3 The minimal-order state observer.

in addition to satisfying (1.76)–(1.78) the observer matrix $D(t)$ is chosen so that (1.75) is asymptotically stable, the effect of any initial observer error $\varepsilon(t_0)$ on the state reconstruction process will diminish exponentially with increasing time $t \geqslant t_0$.

Returning to the constraint Equations (1.76)–(1.78), the number of observer design parameter matrices may be reduced as follows. From (1.78)

$$\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} [V(t), P(t)] = \begin{bmatrix} C(t)V(t) & C(t)P(t) \\ T(t)V(t) & T(t)P(t) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}. \qquad (1.82)$$

Postmultiplication of (1.76) by $P(t)$ and use of (1.82) gives

$$D(t) = T(t)A(t)P(t) + \dot{T}(t)P(t) \qquad (1.83)$$

while postmultiplication of (1.76) by $V(t)$ and use of (1.82) gives

$$E(t) = T(t)A(t)V(t) + \dot{T}(t)V(t). \qquad (1.84)$$

Thus, for a given system description $A(t)$, $B(t)$ and $C(t)$, the observer design matrices may be listed by (1.83), (1.84) and (1.77) as $T(t)$, $P(t)$ and $V(t)$ and the minimal-order observer characterization is complete if we specify the observer initial state vector

$$z(t_0) = T(t_0)x_g \quad \text{for some } x_g \in R^n. \qquad (1.85)$$

A schematic of the minimal-order state observer, represented by Equations (1.71) and (1.72), is given in Fig. 1.3. Notice that the measurements $y(t)$ as well as acting as inputs to the dynamic part of the observer in (1.71) also contribute directly to the state estimate $\hat{x}(t)$ in (1.72). This means that the estimate $\hat{x}(t)$ will be more susceptible to measurement errors in $y(t)$ than in the full-order observer case where filtering of the measurements is achieved through dynamic processing of the measurements. The problem of state estimation in

the presence of measurement noise and system disturbances is considered in Chapter 6. Further redundancy in the number of observer design parameters can be eliminated and will be discussed when we come to consider equivalent classes of minimal-order observers in Chapter 2.

1.7.1 Minimal-order observers for time-invariant systems

The state reconstruction procedure is considerably simplified when, as frequently is the case, the linear continuous-time system is time-invariant. All matrices are independent of time, and the minimal-order observer description (1.77) and (1.83)–(1.85) reduces to

$$D = TAP, \quad E = TAV, \quad G = TB \tag{1.86}$$

$$z(t_0) = Tx_g \quad \text{for some } x_g \in R^n. \tag{1.87}$$

These time-invariant design matrices are more readily instrumented than those of the more general time-varying solution. In particular, T and P may be chosen so that D has only eigenvalues in the left-hand complex plane in order that

$$\dot{\varepsilon}(t) = D\varepsilon(t), \quad \varepsilon(t_0) = z(t_0) - Tx_g \tag{1.88}$$

is asymptotically stable and by (1.81), asymptotic state reconstruction occurs.

1.8 MINIMAL-ORDER OBSERVERS FOR LINEAR DISCRETE-TIME SYSTEMS

Up to now we have considered the state reconstruction problem for continuous-time linear systems. We turn now to consider physical systems whose dynamic behaviour can be modelled by the discrete-time linear vector difference equation

$$x_{k+1} = A_k x_k + B_k u_k \quad k = 0, 1, \ldots \tag{1.89}$$

$$y_k = C_k x_k \tag{1.90}$$

where the system state $x_k \in R^n$, $x_0 \in R^n$ is the initial state, the control input $u_k \in R^r$ and the system output $y_k \in R^m$. Discrete-tme models of the form (1.89) and (1.90) are often the result of sampling of the continuous-time system (1.1) and (1.2).

A minimal-order state observer for the system (1.89) and (1.90) is described by the discrete-time system

$$z_{k+1} = D_k z_k + E_k y_k + G_k u_k \tag{1.91}$$

$$\hat{x}_k = \begin{bmatrix} C_k \\ T_k \end{bmatrix}^{-1} \begin{bmatrix} y_k \\ z_k \end{bmatrix} \tag{1.92}$$

$$= [V_k, P_k] \begin{bmatrix} y_k \\ z_k \end{bmatrix} \tag{1.93}$$

where T_k is chosen such that the above matrix inverse exists. The acceptability of z_k as an approximation for $T_k x_k$ is characterized by the observer reconstruction error

$$\varepsilon_k \triangleq z_k - T_k x_k \tag{1.94}$$

which, by (1.89), (1.90) and (1.91), propagates according to

$$\varepsilon_{k+1} = D_k \varepsilon_k + [D_k T_k + E_k C_k - T_{k+1} A_k] x_k + [G_k - T_{k+1} B_k] u_k. \tag{1.95}$$

Equation (1.95) reduces to the homogeneous equation

$$\varepsilon_{k+1} = D_k \varepsilon_k, \quad k = 0, 1, \ldots \tag{1.96}$$

provided that the observer matrices T_k, D_k, E_k and G_k satisfy

$$T_{k+1} A_k = D_k T_k + E_k C_k \tag{1.97}$$

$$G_k = T_{k+1} B_k. \tag{1.98}$$

Also, from (1.92) and (1.93) we have that

$$\begin{bmatrix} C_k \\ T_k \end{bmatrix}^{-1} = [V_k, P_k]. \tag{1.99}$$

Equations (1.97)–(1.99) are the discrete-time equivalent of the Luenberger constraint Equations (1.76)–(1.78) of the continuous-time treatment. Moreover, the discrete state reconstruction error

$$e_k = \hat{x}_k - x_k \tag{1.100}$$

is given by (1.89) and (1.94) as

$$e_k = P_k \varepsilon_k + (P_k T_k + V_k C_k - I_n) x_k \tag{1.101}$$

which by invoking (1.99) reduces to

$$e_k = P_k \varepsilon_k. \tag{1.102}$$

If the initial state x_0 is known and one sets $z_0 = T_0 x_0$, Equation (1.96) implies that $\varepsilon_k \equiv 0, k = 0, 1, \ldots$ and by (1.102) perfect state reconstruction is achieved. If, as is generally the case, x_0 is unknown one may make a guess for z_0, i.e. set $z_0 = T_0 x_g$ for some $x_g \in R^n$ and then choose the observer matrix D_k so that (1.96) is asymptotically stable.

Now, from (1.99)

$$\begin{bmatrix} C_k \\ T_k \end{bmatrix}[V_k, P_k] = \left[\begin{array}{c|c} C_k V_k & C_k P_k \\ \hline T_k V_k & T_k P_k \end{array}\right] = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}. \tag{1.103}$$

Post-multiplication of (1.97) by P_k and V_k respectively and the use of (1.103) yields

$$D_k = T_{k+1}A_kP_k, \quad E_k = T_{k+1}A_kV_k. \tag{1.104}$$

Thus the discrete-time minimal-order state observer is specified in (1.98) and (1.104) by the matrices T_{k+1}, P_k and V_k and the observer initial state vector

$$z_0 = T_0x_g \quad \text{for some } x_g \in R^n. \tag{1.105}$$

Further redundancy in the number of observer design parameters can be achieved through consideration of an equivalent parametric class of minimal-order observers in Section 2.3. Again, the observer design is considerably simplified when the system is time-invariant in that the observer description reduces to

$$D = TAP, \quad E = TAV, \quad G = TB \tag{1.106}$$

$$z_0 = Tx_g \quad \text{for some } x_g \in R^n. \tag{1.107}$$

Asymptotic state reconstruction may be achieved through choosing T and P so that

$$\varepsilon_{k+1} = TAP\varepsilon_k, \quad \varepsilon_0 = z_0 - Tx_g \tag{1.108}$$

has only eigenvalues with moduli strictly less than unity (inside the unit circle of the complex plane).

1.9 THE DUAL MINIMAL-ORDER OBSERVER

The fundamental property of one system observing another can be applied in the reverse direction to obtain a special kind of controller known as a dual observer [L13]. Just as the function of an observer is to supplement, in a dynamic sense, system information contained in the available system outputs, the role of a dual observer is to supplement that of the available system inputs. Restricting our attention to time-invariant continuous-time linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.109}$$

$$y(t) = Cx(t) \tag{1.110}$$

we have in the special case of $B = I_n$ that the closed-loop system

$$\dot{x}(t) = (A + LC)x(t) \tag{1.111}$$

can be realized by the output feedback control law $u(t) = Ly(t)$. By Theorem (1.16), if the pair (A, C) is completely observable, the eigenvalues of $A + LC$ may be assigned arbitrarily.

In the general case where rank $B = r < n$, only those L which can be factored as $L = BR$ can be implemented with output feedback and the closed-loop system eigenvalues cannot be completely assigned. To realize approximately the response of (1.111), and at the same time achieve complete arbitrary eigenvalue assignment, a dual observer may be used. The dual observer is described by the equations

$$\dot{z}(t) = Dz(t) + Mw(t) \tag{1.112}$$

$$w(t) = y(t) + CSz(t) \tag{1.113}$$

$$u(t) = Jz(t) + Nw(t) \tag{1.114}$$

where the design matrices are constrained to satisfy the relations

$$AS - SD = BJ \tag{1.115}$$

$$L = SM + BN. \tag{1.116}$$

Introducing the new state variable

$$\eta(t) = x(t) + Sz(t) \tag{1.117}$$

the composite system in terms of $\eta(t)$ and $z(t)$ becomes

$$\begin{bmatrix} \dot{\eta}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + LC & O \\ MC & D \end{bmatrix} \begin{bmatrix} \eta(t) \\ z(t) \end{bmatrix}. \tag{1.118}$$

The eigenvalues of the composite system are thus seen to be the eigenvalues of $A + LC$ and the eigenvalues of D and we have the following theorem.

Theorem 1.18 *Corresponding to the nth-order completely controllable and completely observable system (1.109) and (1.110) having r linearly independent inputs, a dynamic dual observer (1.112)–(1.114) of order $n - r$ exists such that the $2n - r$ eigenvalues of the composite system may take any pre-assigned values.*

Notice that the dimension $n - r$ of the minimal-order dual observer will be less than the dimension $n - m$ of the corresponding minimal-order observer for $r > m$. The relations (1.115), (1.116) with $L = I$ are the dual of (1.76), (1.78) for time-invariant systems and can be solved in analogous fashion. Indeed,

Equations (1.115) and (1.116) may be rewritten as

$$[S \quad B]\begin{bmatrix} D \\ J \end{bmatrix} = AS \tag{1.119}$$

$$[S \quad B]\begin{bmatrix} M \\ N \end{bmatrix} = L. \tag{1.120}$$

Since there exists an S such that $[S \quad B]$ is non-singular, define accordingly

$$\begin{bmatrix} U \\ T \end{bmatrix} = [S \quad B]^{-1}. \tag{1.121}$$

Then

$$D = UAS, \quad M = UL \tag{1.122}$$

$$J = TAS, \quad N = TL \tag{1.123}$$

are necessary and sufficient for (1.119) and (1.20) to be satisfied.

1.10 NOTES AND REFERENCES

The notion of reducing the study of an nth order ordinary differential equation to the study of a set of n first-order differential equations (in modern usage, the state equation) was first systematically exploited by Poincaré in his celebrated treatise on celestial mechanics [P6]. In that treatise was born the now familiar idea of representing both past and present dynamical behaviour of a system at any given instant by a set of variables known as the system state variables. A resurgence of interest in the state–space approach to control theory was kindled by Bellman [B6] and Pontryagin [P7] during the mid-1950s in their development of the powerful optimization techniques of dynamic programming and the maximum principle. These and subsequent developments were fostered by the recognition of two things: first, that the state–space approach served as a natural analytical framework for the systematic design of multi-input multi-output systems; and secondly, the growing availability of small special purpose digital computers for information processing and control.

Of the many texts on linear state–space theory or, more generally, linear system theory, the reader may consult [B18], [A8], [R10], [K41], [W10], [K4] and [K18]; see especially Kailath [K4] and Kalman *et al.* [K18] for more detail on what constitutes a finite-dimensional linear dynamical system from two different points of view. The stability Theorem 1.1 and Corollary 1.1 are taken from Kalman and Bertram [K15]. Analogous results for discrete-time linear systems are to be found in the companion paper [K16]. The other standard reference work for Lyapunov's so-called "second method" is Hahn

[H1]; see also Brockett [B18]. The fundamental concept of controllability and its dual concept, observability, were introduced into linear system theory by Kalman [K6]. Also, the closely related but distinct notions (that is for time-varying systems) of reachability and state reconstructability are discussed in Kalman *et al.* [K18] and Willems and Mitter [W5]. The concept of uniform differential observability is due to Silverman and Meadows [S12]. A corresponding discrete treatment of controllability, reachability, observability and state reconstructability is to be found in Weiss [W4] who instead of "reconstructible" uses the term "determinable". The canonical structure theorem (Theorem 1.11) follows Kalman [K10] and Gilbert [G3]. The alternate simpler conditions for controllability of Theorem 1.12 were derived independently by, among others, Popov [P11] and Hautus [H5]; see also Rosenbrock [R10]. Revised treatments and historical remarks on controllability and observability and the duality between them, are to be found in Kalman [K12], [K18] and in [A14]. A numerically efficient algorithm for determining whether a system is controllable, observable, stabilizable, detectable is presented by Davison *et al.* [D4]; see also Paige [P1].

Theorem 1.15 on the equivalence of controllability and arbitrary closed-loop system eigenvalue assignment under state feedback is most often attributed, in its multi-input form, to Wonham [W13]. The result was deduced earlier by Popov [P8] and, for single-input systems, by Rissanen [R5].

The full-order observer or state reconstructor (1.60) of Section 1.6 is a deterministic device and is quite distinct from the Kalman–Bucy filter discussed in Chapter 6. The possibility of a full-order state reconstructor without feedback from the measurements was noted as early as 1958 by Kalman and Bertram [K14]. A full-order device which used measurement feedback was subsequently developed by Kalman [K6] for exactly reconstructing the state of a discrete-time system in a finite number of steps. Theorem 1.16 on the arbitrary adjustment of the dynamics of the state reconstruction process is a central result in linear system theory and is contained somewhat implicitly in the two seminal papers on observers by Luenberger [L9], [L11]. It is perhaps better known in the dual control form (Theorem 1.15) that under state feedback, controllability is equivalent to arbitrary eigenvalue assignment. The separation principle describing the separation of controller and observer design for systems with inaccessible state is also due to Luenberger [L9, L11].

A lucid and well-motivated account of minimal-order state observers for linear time-invariant continuous-time systems is provided in the original paper by Luenberger [L9]. In acknowledgement of Luenberger's prime contribution, and to distinguish it from the full-order Kalman filter, it is often designated the Luenberger observer. There it is assumed that the matrices A and D have no common eigenvalues, thereby guaranteeing a unique T

satisfying the equation $TA - DT = EC$ (see also [L10], [G1] and [P17]). This restriction is removed in the derivation of Equations (1.83) and (1.84) which is based on the time-invariant and time-varying extensions by Newmann [N8] and Yüksel and Bongiorno [Y6] respectively. The time-varying treatments of Tse and Athans [T7] and Yüksel and Bongiorno [Y6] initially postulate that the observer be of arbitrary order $s \geqslant n - m$. The exposition of Section 1.7 is more physically motivated and exploits the fact that the minimum value of s for Equation (1.78) to hold is $s = n - m$. In a more light-hearted vein, the reader is referred to Power [P18] for a lively and entertaining account of the Luenberger observer.

The only prior literature on discrete-time observers for deterministic systems is that of Tse and Athans [T5], [T6]. Our treatment in Section 1.8 is more physically motivated and is developed in analogous fashion to that of the continuous-time study. Mathematically, we would expect the analogy to hold good since the mathematical theory of difference equations is in most respects akin to that of ordinary differential equations.

The notion of a dual observer, due to Brasch and discussed by Luenberger [L13], has been comparatively little studied. Our exposition in Section 1.9 follows that of Luenberger [L13] and is extended to the optimal control of stochastic linear systems by Blanvillain and Johnson [B11], and dynamic deadbeat control of discrete linear systems by Akashi and Imai [A4]. A similar theory may be developed by the methods of Section 1.8 for discrete-time linear time-invariant systems.

Chapter 2

Minimal-order State Observers

2.1 INTRODUCTION

In Chapter 1, it was shown that the n-dimensional state vector of a completely observable linear finite-dimensional system with m independent outputs can be reconstructed by an observer of order $n - m$ which is itself a linear finite-dimensional dynamic system. This minimal-order state observer will generate an asymptotic state estimate provided the coefficient matrix $D(t)$ is stable and the design matrices satisfy a constraint equation (Equation (1.78)). The present chapter continues the discussion by reducing the inherent redundancy in the observer description through exhibiting the original system in various state–space realizations in which the system structure is considerably simplified.

By transforming the system to a co-ordinate basis where the system outputs yield directly m of the system state variables, as in Section 2.2, it is possible to specify the minimal-order observer by a single arbitrary gain parameter matrix. In Section 2.3 the definition of this "parametric" class of observers brings out more explicitly the structural relationship that exists between an observer and the "observed" system. Section 2.4 develops further the equivalence between the idea of system state reconstruction and the existence of an asymptotically stable "parametric" observer and how they both depend upon the observability of a reduced-order pair $(A_{22}(t), A_{12}(t))$. Simple design methods for choosing the observer gain matrix to ensure the asymptotic stability of the observer for both time-varying and time-invariant systems are presented.

Alternative design methods in Sections 2.5 and 2.6 are based on the approach, originally suggested by Luenberger [L11] of first transforming the system to a multi-output observable companion form. In this form, the system effectively decomposes into m single-output subsystems for each of which a sub-observer may be designed. The composite observer is again of dimension $n - m$ although the observer eigenvalues may only be arbitrarily assigned

through arbitrary eigenvalue assignment within each of the m lower dimensional sub-observers.

2.2 AN EQUIVALENT CLASS OF LINEAR SYSTEMS

Consider the continuous-time linear finite-dimensional system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.1}$$

$$y(t) = C(t)x(t) \tag{2.2}$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$ and the observation matrix $C(t)$ is assumed to be of full rank $m < n$. Little loss of generality is incurred by this assumption since $C(t)$ can frequently be reduced to a matrix of constant full rank by the elimination of any redundant (linearly dependent) outputs in the measurement vector $y(t)$.

In the development, it is extremely convenient to transform the system realization (2.1) and (2.2) by way of the invertible state transformation

$$\bar{x}(t) = M(t)x(t) \tag{2.3}$$

to the equivalent state–space realization

$$\dot{\bar{x}}(t) = (MAM^{-1} + \dot{M}M^{-1})\bar{x}(t) + MBu(t) \tag{2.4}$$

$$y(t) = CM^{-1}\bar{x}(t) = [I_m, 0]\bar{x}(t). \tag{2.5}$$

Many physical system realizations take this form. Those which do not may be readily transformed if one sets

$$M(t) = \begin{bmatrix} C(t) \\ S(t) \end{bmatrix} \tag{2.6}$$

where $S(t)$ is any $(n - m) \times m$ matrix chosen to make $M(t)$ non-singular. Equation (2.5) follows from (2.2) and the upper partition of

$$\left[\frac{C(t)}{S(t)}\right] M(t)^{-1} = \left[\frac{I_m, O}{O, I_{n-m}}\right]. \tag{2.7}$$

If, upon collection of linearly independent columns, $C(t)$ may be partitioned as

$$C(t) = [C_1(t), C_2(t)] \tag{2.8}$$

where $C_1(t)$ is an $m \times m$ non-singular matrix, a particularly convenient choice of $S(t)$ is

$$S(t) = [O \quad I_{n-m}]. \tag{2.9}$$

Thus, it is henceforth assumed *without loss of generality* that the continuous-

time linear system is in the equivalent form (2.4) and (2.5) or in the partitioned form

$$\dot{x}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} x(t) + \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} u(t) \tag{2.10}$$

$$y(t) = [I_m, O]x(t). \tag{2.11}$$

Similarly, the discrete-time linear system (1.27) and (1.28) is assumed to be in the form

$$x_{k+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_k x_k + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}_k u_k \tag{2.12}$$

$$y_k = [I_m, O]x_k. \tag{2.13}$$

This equivalent class of systems where the measurements yield the first m system state variables directly lends itself to a more explicit characterization of minimal-order state observers than hitherto achieved.

2.3 OBSERVER PARAMETERIZATION

In Section 1.7 we established that the continuous-time minimal-order observer coefficient matrices (1.83), (1.84) and (1.77) are specified by the design matrices $T(t)$, $P(t)$ and $V(t)$ which satisfy the constraint Equation (1.78) or

$$\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} [V(t), P(t)] = I_n. \tag{2.14}$$

Further redundancy in characterizing the observer (1.71) and (1.72) by $T(t)$, $P(t)$ and $V(t)$ may be eliminated by assuming that the system is in the state–space realization (2.10) and (2.11) where

$$C(t) = [I_m, O] \tag{2.15}$$

and partitioning $V(t)$ and $P(t)$ according to

$$[V(t), P(t)] = \begin{bmatrix} V_1(t) & P_1(t) \\ V_2(t) & P_2(t) \end{bmatrix}. \tag{2.16}$$

Substitution of (2.15) and (2.16) into (2.14) yields

$$V_1(t) = I_m, \quad P_1(t) = O_{n-m,m} \tag{2.17}$$

so that

$$[V(t), P(t)] = \begin{bmatrix} I_m & O \\ V_2(t) & P_2(t) \end{bmatrix} \tag{2.18}$$

which is non-singular if, and only if, $P_2(t)$ is non-singular. Now $[V(t), P(t)]$ in (2.18) is of a simpler structure than that of (2.16) and is readily inverted to give

$$\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} = [P(t), V(t)]^{-1} = \begin{bmatrix} I_m & O \\ -P_2^{-1}(t)V_2(t) & P_2^{-1}(t) \end{bmatrix} \tag{2.19}$$

whence

$$T(t) = P_2^{-1}(t)[-V_2(t), I_{n-m}]. \tag{2.20}$$

At this point it is clear from (2.18) and (2.20), that the observer matrices are completely specified by $P_2(t)$ and $V_2(t)$. That is, substitution of (2.18) and (2.20) in (1.83), (1.84) and (1.77) and use of the identity

$$\dot{P}_2^{-1}(t) \equiv -P_2^{-1}(t)\dot{P}_2(t)P_2^{-1}(t) \tag{2.21}$$

results in

$$D(t) = P_2^{-1}(t)[A_{22}(t) - V_2(t)A_{12}(t)]P_2(t) - P_2^{-1}(t)\dot{P}_2(t) \tag{2.22}$$

$$E(t) = P_2^{-1}(t)[A_{21}(t) - V_2(t)A_{11}(t) + A_{22}(t)V_2(t) - V_2(t)A_{12}(t)V_2(t) - \dot{V}_2(t)] \tag{2.23}$$

$$G(t) = P_2^{-1}(t)[B_2(t) - V_2(t)B_1(t)]. \tag{2.24}$$

Also, directly from (2.18) we have

$$P(t) = \begin{bmatrix} O \\ P_2(t) \end{bmatrix}, \quad V(t) = \begin{bmatrix} I_m \\ V_2(t) \end{bmatrix}. \tag{2.25}$$

Summarizing thus far, we have obtained an equivalent class of observers for the system (2.10) and (2.11) called the "parametric class"* of observers that is completely specified by the arbitrary gain parameter matrix $V_2(t)$. Inspection of (2.22) indicates that the asymptotic stability of the error system (1.75) depends only on the choice of $V_2(t)$ when $P_2(t)$ is chosen to be any constant non-singular matrix. In this case, the stability of $D(t)$ is identical to that of

$$\hat{D}(t) = A_{22}(t) - V_2(t)A_{12}(t) \tag{2.26}$$

and with

$$\hat{\varepsilon}(t) = P_2\varepsilon(t) \tag{2.27}$$

the observer error Equation (1.75) becomes

$$\dot{\hat{\varepsilon}}(t) = [A_{22}(t) - V_2(t)A_{12}(t)]\hat{\varepsilon}(t). \tag{2.28}$$

No loss of generality is incurred if we take the non-singular matrix P_2 to be the identity matrix since one can always redefine the observer state vector as

* A number of authors [Y6], [R6], [M9], [O25] including the writer have in the past rather loosely referred to this class as the "canonical class" of observers. In view of the precise mathematical meaning that the term "canonical form" has in the literature (see Notes and References 2.7), we adopt instead the description "parametric class".

$$\hat{z}(t) = P_2(t)z(t). \tag{2.29}$$

It is readily seen from the parametric observer structure that the measurements $y(t)$ contribute directly to the state estimate while the unknown part of the system state vector is "modelled" by the observer state vector $z(t)$. We shall see in Section 2.4 that any reconstruction error, due to imperfect "modelling" of the initial state, can be reduced asymptotically by choosing $V_2(t)$ so that (2.22) or (2.26) is a stability matrix.

Also, we shall examine more precisely the role played by observability in the state reconstruction process. Observability and state reconstructability, defined in Section 1.3, refer to the possibility of determining the present system state from future or past output measurements respectively. In the context of a parametric class of observers the following definition is appropriate.

Definition 2.1 The system (2.10) and (2.11) is asymptotically state reconstructible if there exists an observer

$$\begin{aligned}\dot{z}(t) &= (A_{22}(t) - V_2(t)A_{12}(t))z(t) + (A_{21}(t) - V_2(t)A_{11}(t)\\ &\quad + A_{22}(t)V_2(t) - V_2(t)A_{12}(t)V_2(t) - \dot{V}_2(t))y(t)\\ &\quad + (B_2(t) - V_2(t)B_1(t))u(t)\end{aligned} \tag{2.30}$$

$$\hat{x}(t) = \begin{bmatrix} O \\ I_{n-m} \end{bmatrix} z(t) + \begin{bmatrix} I_m \\ V_2(t) \end{bmatrix} y(t)$$

$$z(t_0) = [-V_2(t_0), I_{n-m}]x_g \quad \text{for any } x_g \in R^n$$

such that

$$\lim_{t\to\infty} (\hat{x}(t) - x(t)) = O. \tag{2.31}$$

2.3.1 Discrete-time parametric observers

It is seen in Section 1.8 that the theory of discrete-time minimal-order observers is in many ways analogous to the continuous-time theory. Our expectation (and hope) that the analogy extends to the equivalent class of parametric observers is happily satisfied. Assuming the discrete-time system to be in the form (2.12) and (2.13), relations similar to (2.17)–(2.20) hold for V_k, P_k and T_k. In other words,

$$P_k = \begin{bmatrix} O \\ P_2^k \end{bmatrix}, \quad V_k = \begin{bmatrix} I_m \\ V_2^k \end{bmatrix} \tag{2.32}$$

and

$$T_k = P_2^{k^{-1}}[-V_2^k, I_{n-m}]. \tag{2.33}$$

Since no loss of generality is incurred if one assumes P_2^k to be the identity

matrix, substitution of (2.32) and (2.33) into (1.104) and (1.98) gives rise to a parametric class of observers for linear discrete-time systems. Again, this parametric class of observers is explicitly specified by a single gain matrix V_2^k and for the purposes of asymptotic state reconstruction may be defined as follows:

Definition 2.2 The system (2.12) and (2.13) is asymptotically state reconstructible if there exists an observer

$$\begin{aligned} z_{k+1} &= (A_{22}^k - V_2^{k+1}A_{12}^k)z_k + (A_{21}^k + A_{22}^k V_2^k - V_2^{k+1}(A_{11}^k + A_{12}^k V_2^k))y_k \\ &\quad + (B_2^k - V_2^{k+1}B_1^k)u_k \\ \hat{x}_k &= \begin{bmatrix} 0 \\ I_{n-n} \end{bmatrix} z_k + \begin{bmatrix} I_m \\ V_2^k \end{bmatrix} y_k \qquad (2.34) \\ z_0 &= [-V_2^0, I_{n-m}]x_g \quad \text{for any } x_g \in R^n \end{aligned}$$

such that

$$\lim_{k\to\infty} (\hat{x}_k - x_k) = 0. \qquad (2.35)$$

2.4 PARAMETRIC OBSERVER DESIGN METHODS

Given that the minimal-order observer may be explicitly parameterized by a single arbitrary gain matrix V_2, various design objectives can be met through different choices of V_2. Of the several different design criteria, optimal observer design in an integral least-square reconstruction error sense is considered in Chapter 4; the state reconstruction of discrete-time linear systems in a minimum number of time steps, also known as deadbeat observer design, is the subject of Chapter 5; Chapter 6 treats the design of observers that are optimal in a linear least-square error sense for stochastic systems.

A common feature of all these design methods for linear time-invariant systems is that the choice of V_2 essentially shifts the eigenvalues of the observer coefficient matrix to desired (stable) positions in the complex plane. We initially consider parametric observer design for asymptotic state reconstruction for linear time-varying systems.

2.4.1 Asymptotic state reconstruction

Bearing Definition 2.1 in mind, we have conditions for uniform state reconstruction in the following.

Theorem 2.1 *A necessary and sufficient condition for the state reconstructability of the system (2.10) and (2.11) is that the observer (2.30) be uniformly asymptotically stable.*

Proof (*sufficiency*) From (1.81) and (2.25) with $P_2 \equiv I_{n-m}$, the state estimation error $e(t) \triangleq \hat{x}(t) - x(t)$ is given by

$$e(t) = P(t)\varepsilon(t) = \begin{bmatrix} 0 \\ \varepsilon(t) \end{bmatrix} \tag{2.36}$$

where, by (2.28), $\varepsilon(t)$ satisfies

$$\dot{\varepsilon}(t) = (A_{22}(t) - V_2(t)A_{12}(t))\varepsilon(t). \tag{2.37}$$

Now if the observer (2.30) is uniformly asymptotically stable, (2.37) implies that

$$\lim_{t\to\infty} \varepsilon(t) = 0. \tag{2.38}$$

Hence from (2.36)

$$\lim_{t\to\infty} e(t) = 0 \tag{2.39}$$

and by Definition 2.1 the system is state reconstructible. The *necessity* of uniform asymptotic stability is obvious by a converse argument. Q.E.D. □

Theorem 2.2 *If $(A_{22}(t), A_{12}(t))$ is uniformly completely observable there exists a uniformly asymptotically stable minimal-order observer (2.30)). Furthermore, the state of the system (2.10) and (2.11) may be reconstructed in a uniform asymptotic manner.*

Proof From Theorem 1.1 the proof rests upon generating a suitable (quadratic) Lyapunov function. Choosing

$$\dot{V}_L(\varepsilon(t), t) = -\varepsilon'(t)Q(t)\varepsilon(t) \tag{2.40}$$

where $Q(t)$ is a real symmetric positive definite matrix it follows from (2.37) that

$$V_L(\varepsilon(t), t) = \varepsilon'(t)P_L(t)\varepsilon(t) \tag{2.41}$$

where $P_L(t)$ satisfies

$$\begin{aligned} -\dot{P}_L(t) = {} & (A_{22}(t) - V_2(t)A_{12}(t))'P_L(t) + P_L(t) \\ & \times (A_{22}(t) - V_2(t)A_{12}(t)) + Q(t). \end{aligned} \tag{2.42}$$

If we further select the observer gain $V_2(t)$ such that

$$V_2(t) = \tfrac{1}{2}P_L^{-1}(t)A'_{12}(t)M(t) \tag{2.43}$$

where $M(t)$ is an arbitrary real symmetric positive definite matrix, then substitution of (2.43) into (2.42) yields

$$\dot{P}_L(t) = -P_L(t)A_{22}(t) - A'_{22}(t)P_L(t) - Q(t) + A'_{12}(t)M(t)A_{12}(t). \tag{2.44}$$

Since (2.44) is linear in $P_L(t)$ it possesses the unique solution

$$P_L(t) = \phi'(t_0, t)P_L(t_0)\phi(t_0, t) - \int_{t_0}^{t} \phi'(\lambda, t)Q(\lambda)\phi(\lambda, t)\, d\lambda$$
$$+ \int_{t_0}^{t} \phi(\lambda, t)A'_{12}(\lambda)M(\lambda)A_{12}(\lambda)\phi(\lambda, t)\, d\lambda \tag{2.45}$$

where $\phi(t, t_0)$ is the state transition matrix of the free system

$$\dot{x}_2(t) = A_{22}(t)x_2(t). \tag{2.46}$$

Now uniform complete observability of the pair $(A_{22}(t), A_{12}(t))$ is defined in Theorem 1.6 by the existence of positive constants α_1, α_2 and τ such that

$$0 < \alpha_1 I_n \leqslant \int_{t}^{\tau+t} \phi'(\lambda, \tau + t)A'_{12}(\lambda)A_{12}(\lambda)\phi(\lambda, \tau + t)\, d\lambda$$
$$\leqslant \alpha_2 I_n \tag{2.47}$$

for all t.

Therefore, for any choice of $Q(t) > 0$, there exists, by (2.45) and (2.47), an $M(t) > 0$ and $P_L(t_0) > 0$ such that $P_L(t)$ will be real symmetric and positive definite for all $t \geqslant t_0$. Consequently, there exists an observer (2.30) which is uniformly asymptotically stable in the sense of Lyapunov (Theorem 1.1) if $(A_{22}(t), A_{12}(t))$ is uniformly completely observable. The last statement of Theorem 2.2 follows from Theorem 2.1. Q.E.D. □

The above proof is a constructive one in that it also suggests a design procedure for choosing the observer gain matrix $V_2(t)$ so that the observer (2.30) is uniformly asymptotically stable. Uniform asymptotic stability of the observer (2.30) ensures that it will remain stable in the presence of random disturbances due to uncertainties in the system model (2.10) and (2.11). This is an important practical advantage in the design of observers as state reconstructors. It is shown in Lemma 2.1 that, at any instant t, observability of the pair $(A(t), C(t))$ is sufficient to ensure observability of the pair $(A_{22}(t), A_{12}(t))$. Attempts to extend the approach of Theorem 2.2 to discrete-time systems whereby the stabilizing observer gain is specified by an expression analogous to (2.43), have so far proved unfruitful. In the special case where the system is time-invariant, the stabilizing observer gain is given by

$$V_2 = \tfrac{1}{2} P_L^{-1} A'_{12} M$$

where P_L is the unique symmetric positive-definite solution of

$$0 = -P_L A_{22} - A'_{22} P_L - Q + A'_{12} M A_{12}.$$

A generalization of this case is to specify observer stability to the left of the $-\alpha$ line in the complex plane for any $\alpha \geqslant 0$ [A8] where V_2 is as before but P_L is the unique symmetric positive-definite solution of

$$0 = -P_L(A_{22} - \alpha I_{n-m}) + (A_{22} - \alpha I_{n-m})' P_L - Q + A'_{12} M A_{12}.$$

2.4.2 Parametric observer eigenvalue assignment

For time-invariant systems the asymptotic state reconstruction is equivalent to the provision that all the eigenvalues of the observer coefficient matrix $D = A_{22} - V_2 A_{12}$ lie in the left-hand side of the complex plane, assuming any complex eigenvalues only occur in conjugate pairs. Preliminary to any observer eigenvalue assignment for asymptotic error reduction in (2.37), we have immediately from Theorem 1.16 the useful result.

Theorem 2.3 *Corresponding to the real matrices A_{22} and A_{12} there is a real matrix V_2 such that the set of eigenvalues of $A_{22} - V_2 A_{12}$ can be arbitrarily assigned (subject to complex pairing) if and only if (A_{22}, A_{12}) is a completely observable pair.*

The question of observability of the pair (A_{22}, A_{12}) is answered by the following lemma.

Lemma 2.1 *If the pair (A, C) is completely observable, then so is the pair (A_{22}, A_{12}).*

Proof If (A, C) is completely observable, (1.33) implies

$$\operatorname{rank}\,[C', A'C', \ldots A'^{n-1}C'] = n \tag{2.48}$$

or for the time-invariant system of the form (2.10) and (2.11)

$$\operatorname{rank}\begin{bmatrix} I_m & 0 \\ A_{11} & A_{12} \\ A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} = n. \tag{2.49}$$

Since the rank of the matrix in (2.49) is unaltered by elementary row operations, one has that

$$\operatorname{rank}\begin{bmatrix} I_m & 0 \\ \cdots & A_{12} \\ \cdots & A_{12}A_{22} \\ \cdots & \cdots \\ \cdots & A_{12}A_{22}^{n-1} \end{bmatrix} = n \tag{2.50}$$

where the third row partition of the new matrix is the third row partition of the old matrix $-A_{11}^2$ (first row) $-A_{11}$ (second row), etc. Noting that rank $I_m = m$ and that, by the Cayley-Hamilton theorem (Appendix A), all powers of the $(n - m) \times (n - m)$ matrix A_{22} greater than $n - m - 1$ are linearly dependent on powers 0 to $n - m - 1$, it follows that

$$\operatorname{rank}\begin{bmatrix} A_{12} \\ A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^{n-m-1} \end{bmatrix} = n - m \tag{2.51}$$

or (A_{22}, A_{12}) is completely observable. Q.E.D. □

Assuming the pair (A_{22}, A_{12}) to be completely observable and hence, by Theorem 2.3, the possibility of arbitrary eigenvalue assignment, there remains the practical problem of constructing the matrix V_2 so that $D = A_{22} - V_2A_{12}$ is a stability matrix.

A particularly attractive eigenvalue assignment procedure, both from the point of view of simplicity and the amount of freedom left to the designer, is the "unity rank" method suggested by Gopinath [G4]. Basically, this method consists of constraining the matrix V_2 to be of unity rank by defining it in the dyadic form

$$V_2 = k\alpha' \tag{2.52}$$

where $k \in R^{n-m}$ is a vector of gains to be chosen to assign the eigenvalues of $A_{22} - V_2A_{12}$ to desired stable locations in the complex plane and $\alpha \in R^m$ is a vector to be specified at the designer's discretion, possibly to meet other observer design requirements such as sensitivity reduction to modelling errors, etc. An appropriate choice of k, and hence V_2, is provided by the following theorem.

Theorem 2.4 *If the characteristic polynomial of A_{22} is*

$$\det(sI_{n-m} - A_{22}) = s^{n-m} + \sum_{i=1}^{n-m} r_i s^{n-m-i}$$

and the characteristic polynomial that specifies the desired *stable eigenvalue locations of $A_{22} - V_2A_{12}$ is*

$$\det(sI_{n-m} - (A_{22} - V_2A_{12})) = s^{n-m} + \sum_{i=1}^{n-m} \gamma_i s^{n-m-i}$$

then, a choice of V_2 that accomplishes this desired eigenvalue assignment is given by

$$V_2 = \begin{bmatrix} \alpha' A_{12} \\ \alpha' A_{12} A_{22} \\ \vdots \\ \alpha' A_{12} A_{22}^{n-m-1} \end{bmatrix}^{-1} R^{-1}[\gamma - r]\alpha' \tag{2.53}$$

where $r = [r_1, r_2, \ldots r_{n-m}]'$ and $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_{n-m}]'$ are vectors of the coefficients of the characteristic polynomials of A_{22} and $A_{22} - V_2A_{12}$ respectively, and

$$R = \begin{bmatrix} 1 & 0 & \cdots & & 0 & 0 \\ r_1 & 1 & \cdots & & 0 & 0 \\ r_2 & r_1 & \cdots & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ r_{n-m-2} & r_{n-m-3} & \cdots & r_1 & 1 & 0 \\ r_{n-m-1} & r_{n-m-2} & \cdots & r_2 & r_1 & 1 \end{bmatrix}$$

Moreover, the vector $\alpha \in R^m$ will almost always exist.

Proof The proof of Theorem 2.4 is lengthy but straightforward and the interested reader is referred to Gopinath [G4]. An alternative proof using frequency-domain arguments is presented by Young and Willems [Y5]. □

2.5 THE OBSERVABLE COMPANION FORM AND THE LUENBERGER OBSERVER

We have seen that it is a decided advantage to be able to transform the linear system (2.1) and (2.2) to another state–space realization of simpler structure. Of course, the choice of transformation matrix M such as that of (2.6) is not unique. Another convenient structured form of (2.1) and (2.2) is the so-called observable companion form [L12] for which, to begin with, we shall consider for single-output time-invariant systems only.

Consider the single-output time-invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.54}$$

$$y(t) = cx(t) \tag{2.55}$$

which by (1.33) is observable in the sense that the rank of the observability matrix $W = [c', A'c', \ldots A'^{n-1}c']$ is n. Then, the system (2.54) and (2.55) may be transformed to the observable companion form

$$\dot{\bar{x}}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ \vdots & 1 & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \bar{x}(t) + M^{-1}Bu(t) \tag{2.56}$$

$$y(t) = [0 \quad 0 \quad \cdots \quad 1]\bar{x} \tag{2.57}$$

by way of the non-singular state transformation

$$x(t) = M\bar{x}(t) \tag{2.58}$$

where

$$M = [g_1, Ag_1, \ldots, A^{n-1}g_1] \tag{2.59}$$

and $g_1 \in R^n$ is set equal to the last (nth) column of the inverse U^{-1} of the observability matrix

$$U = \begin{bmatrix} c \\ cA \\ cA^{n-1} \end{bmatrix} \tag{2.60}$$

One immediate benefit of transforming the system (2.54) and (2.55) to a

structurally simpler observable canonical form lies in the determination of the characteristic polynomial $\det(sI - A)$ of A. That is, if

$$\bar{A} \triangleq M^{-1}AM = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & 1 & -a_{n-1} \end{bmatrix} \tag{2.61}$$

the characteristic polynomial of A is given by

$$\det(sI - A) = \det(MsIM^{-1} - M\bar{A}M^{-1}) = \det(M)\det(sI - \bar{A})\det(M^{-1})$$

$$\det(sI - \bar{A}) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{2.62}$$

Consider now the application of the full-order observer (1.60) to the system (2.56) and (2.57) whereby the time evolution of the state reconstruction error is determined by the eigenvalues of the observer coefficient matrix $\bar{A} - g\bar{c}$ of (1.62). Since the pair $(\bar{A}, \bar{c})$ is observable, we have by Theorem 1.16 that the eigenvalues of $\bar{A} - g\bar{c}$ may be chosen arbitrarily and so set equal to the roots of the characteristic polynomial $s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0$. Thus, if one sets

$$\bar{A} - g\bar{c} = \begin{bmatrix} 0 & \cdots & 0 & -\beta_0 \\ 1 & & & -\beta_1 \\ \vdots & 1 & & \vdots \\ 0 & \cdots & 1 & -\beta_{n-1} \end{bmatrix} \tag{2.63}$$

it is immediately apparent from the structure of (2.56) and (2.57) that

$$g = \begin{bmatrix} \beta_0 - a_0 \\ \beta_1 - a_1 \\ \beta_{n-1} - a_{n-1} \end{bmatrix} \tag{2.64}$$

and the full nth-order observer (1.60) is completely specified.

Alternatively, since we already know the last state variable $\bar{x}_n(t)$ from (2.57) it makes sense to design an observer with arbitrary dynamics of order $n - 1$ to reconstruct the remaining $n - 1$ state variables of (2.56) and (2.57). Indeed, from our earlier discussion on parametric observers we have after appropriate partitioning of (2.56) and (2.57) that the $(n - 1)$th order observer may be

characterized by

$$D = A_{11} - V_1 A_{21}$$
$$= \begin{bmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix} - V_1[0 \cdots 01]. \tag{2.65}$$

Thus, if we take

$$D = \begin{bmatrix} 0 & & & -\beta_0 \\ 0 & 1 & & -\beta_1 \\ & & \ddots & \\ 0 & & 1 & -\beta_{n-2} \end{bmatrix} \tag{2.66}$$

it is clear from comparing (2.65) and (2.66) that

$$V_1 = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_{n-2} \end{bmatrix} \tag{2.67}$$

and similar to (2.30) the $(n-1)$th order observer is represented by

$$\dot{z}(t) = \begin{bmatrix} 0 & & \cdots & -\beta_0 \\ 0 & & 1 & -\beta_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & -\beta_{n-2} \end{bmatrix} z(t)$$
$$+ \left[\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-2} \end{bmatrix} (a_{n-1} - \beta_{n-2}) + \begin{bmatrix} & -a_0 \\ \beta_0 & -a_1 \\ \beta_1 & -a_2 \\ & \vdots \\ \beta_{n-3} & -a_{n-2} \end{bmatrix} \right] y + TM^{-1}Bu \tag{2.68}$$

$$\hat{x}(t) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ & \ddots & \\ & 0 & 1 \\ 0 & \cdots & 0 \end{bmatrix} z(t) + \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_{n-2} \\ 1 \end{bmatrix} y(t). \tag{2.69}$$

Again, since the β_i corresponding to any choice of V_1 in (2.67) are arbitrary, the $n-1$ eigenvalues of the $(n-1)$th order observer (2.68) and (2.69) can be chosen arbitrarily.

This single output result can now be extended to the general multi-output case by transforming the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.70}$$

$$y(t) = Cx(t) \tag{2.71}$$

in accordance with the invertible state transformation

$$x(t) = M\bar{x}(t) \tag{2.72}$$

where
$$M = [g_1, Ag_1, \ldots, A^{d_1-1}g_1, g_2, Ag_2, \ldots, A^{d_m-1}g_m] \tag{2.73}$$

and each $g_i \in R^n$ is set equal to the σ_ith column of the inverse V^{-1} of the row-reordered observability matrix

$$V = \begin{bmatrix} c_1 \\ c_1A \\ \vdots \\ c_1A^{d_1-1} \\ \hdashline c_2 \\ \vdots \\ c_2A^{d_2-1} \\ \hdashline \vdots \\ c_mA^{d_m-1} \end{bmatrix} \tag{2.74}$$

The n linearly independent rows of V are selected from the observability matrix of (1.33) according to Luenberger [L12] where at any stage of the selection procedure c_jA^k is linearly independent of all lower powers of c_j times A, otherwise it is omitted. The m integers d_i are known as the *observability indices* of the system of which max $\{d_i\}$, $i = 1, 2, \ldots, m$, is defined as the *observability index* ν of the system (see also Equation (1.34)); also

$$\sum_{i=1}^{m} d_i = n, \quad \text{the system dimension.}$$

Then the system in its multivariable observable companion form is described by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + M^{-1}Bu(t) \tag{2.75}$$

$$y(t) = \bar{C}\bar{x}(t) \tag{2.76}$$

where $\bar{A} = M^{-1}AM$ and $\bar{C} = CM$ respectively have the following structure.

$$\bar{A} = \begin{array}{c}
\left[\begin{array}{ccccc|ccccc|c|ccccc}
0 & 0 & \cdots & 0 & x & 0 & 0 & \cdots & & x & & 0 & 0 & \cdots & & x \\
1 & 0 & & 0 & x & 0 & 0 & \cdots & & x & & 0 & 0 & \cdots & & x \\
0 & 1 & \cdots & 0 & x & & & & & & \cdots & & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & & \vdots & & \vdots & \vdots & & \vdots & \\
0 & 0 & \cdots & 1 & x & 0 & 0 & \cdots & & x & & 0 & 0 & \cdots & & x \\
\hline
0 & 0 & \cdots & & x & 0 & 0 & \cdots & 0 & x & & 0 & 0 & \cdots & & x \\
0 & 0 & \cdots & & x & 1 & 0 & \cdots & 0 & x & & 0 & 0 & \cdots & & x \\
 & & & & & 0 & 1 & \cdots & 0 & x & \cdots & & & & & \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\
0 & 0 & \cdots & & x & 0 & 0 & \cdots & 1 & x & & 0 & 0 & \cdots & & x \\
\hline
 & & \vdots & & & & & \vdots & & & & & & \vdots & & \\
\hline
0 & 0 & \cdots & & x & 0 & 0 & \cdots & & x & & 0 & 0 & \cdots & 0 & x \\
0 & 0 & \cdots & & x & 0 & 0 & \cdots & & x & & 1 & 0 & \cdots & 0 & x \\
 & & & & & & & & & & \cdots & 0 & 1 & \cdots & 0 & x \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & & x & 0 & 0 & \cdots & & x & & 0 & 0 & \cdots & 1 & x
\end{array}\right]
\begin{array}{c}
\\ \\ d_1 \\ \\ \\ \\ \\ d_2 \\ \\ \\ \\ \\ \\ d_m \\ \\ \\
\end{array} \\
\begin{array}{ccc}
\uparrow \qquad\qquad\qquad & \uparrow \qquad\qquad\qquad & \uparrow \\
\sigma_1 \qquad\qquad\qquad & \sigma_2 \qquad\qquad\qquad & \sigma_m
\end{array}
\end{array} \tag{2.77}$$

and

$$\bar{C} = \begin{array}{c}
\left[\begin{array}{ccccc|c|ccccc}
0 & 0 & \cdots & 0 & 1 & & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & x & & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & x & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & x & & 0 & 0 & \cdots & 0 & 1
\end{array}\right] \\
\begin{array}{cc}
\longleftarrow d_1 \longrightarrow \qquad\qquad & \longleftarrow d_m \longrightarrow \\
\qquad\quad \uparrow \qquad\qquad & \qquad\qquad \uparrow \\
\qquad\quad \sigma_1 \qquad\qquad & \qquad\qquad \sigma_m (= n)
\end{array}
\end{array} \tag{2.78}$$

Notice that $\bar{A}$ consists of m observable companion forms, each of dimension d_i, along the leading diagonal of which the non-trivial columns are denoted by

σ_i. The companion form of largest dimension is of dimension ν, the observability index of the system defined in (1.34).

Let us now consider the ith subsystem of (2.75)–(2.78), $i = 1, 2, \ldots, m$, described by the state vector

$$\bar{x}_i(t) = \begin{bmatrix} \bar{x}_{\sigma_{i-1}+1}(t) \\ \bar{x}_{\sigma_i}(t) \end{bmatrix} \in R^{d_i} \tag{2.79}$$

which in view of the block decomposition of (2.77) satisfies

$$\dot{\bar{x}}_i(t) = \bar{A}_{ii}\bar{x}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \bar{A}_{ij}\bar{x}_j(t) + [M^{-1}B]_i u(t) \quad i = 1, 2, \ldots, m \tag{2.80}$$

where $\bar{A}_{ii} \in R^{d_i \times d_i}$ and $\bar{A}_{ij} \in R^{d_i \times d_j}$ denote the leading diagonal and off-diagonal blocks of (2.77) respectively and $[M^{-1}B]_i \in R^{d_i \times r}$ is the ith submatrix of $M^{-1}B$. Also, let $\bar{C}_m$ be the $m \times m$ lower triangular matrix formed from $\bar{C}$ in (2.78) by eliminating all but the m non-trivial columns of $\bar{C}$ so that we have the relation

$$\bar{C} = \bar{C}_m \begin{array}{c} \left[\begin{array}{ccccc|c|ccccc} 0 & 0 & \cdots & 0 & 1 & & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} & & & & \uparrow \qquad\qquad\qquad\qquad\qquad \uparrow \\ & & & & \sigma_1 \qquad\qquad\qquad\qquad \sigma_m(=n) \end{array} \end{array} \tag{2.81}$$

Then, using (2.76) and (2.81), a new output equation may be derived as

$$\begin{aligned} \bar{y}(t) &= \bar{C}_m^{-1} y(t) \\ &= \left[\begin{array}{ccccc|c|ccccc} 0 & 0 & \cdots & 0 & 1 & & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 & & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \bar{x}(t) \\ &= \begin{bmatrix} \bar{x}_{\sigma_1} \\ \bar{x}_{\sigma_2} \\ \vdots \\ \bar{x}_{\sigma_m} \end{bmatrix} \end{aligned} \tag{2.82}$$

so that the $\bar{x}_{\sigma_i}$, $i = 1, 2, \ldots, m$, components of $\bar{x}$ are available. Since the first $d_i - 1$ columns of $\bar{A}_{ij}$ are identically zero, the second term on the right-hand side

of (2.80) may be expressed in terms of these $m - 1$ measurable components $\bar{x}_{\sigma_j}$ $(j \neq i)$. Thus it is apparent that the ith subsystem described by (2.80) is a d_i-dimensional single-output system in observable companion form driven by the directly measurable system signals $u(t)$ and $\bar{x}_{\sigma_j}$ $(j \neq i)$. Therefore, an observer of the type (2.68) and (2.69) of order $d_i - 1$ can be employed for each of the m subsystems. That is, an observer of total dimension $\sum_{i=1}^{m} (d_i - 1) = n - m$ can be used to estimate the entire state vector of (2.75). Furthermore, the $n - m$ eigenvalues of the observer are arbitrary through arbitrary eigenvalue assignment of each of the m subobservers.

2.6 TIME-VARYING COMPANION FORMS AND OBSERVER DESIGN

It may be wondered if the observer design based on a system in Luenberger-type companion form may be extended to time-varying linear systems. If so, a second question might concern the practicality of instrumenting any observer based on the many time-varying companion-form parameters. The answer to the first question is in the affirmative, while the answer to the second question is a qualified yes if the time-varying system companion-form somehow yields a *time-invariant* observer coefficient matrix. Consequently, an observer error stabilization method similar to the eigenvalue assignment procedure employed in Section 2.5 should be feasible.

The key to this development again lies in appropriate system transformation to companion form, this time via the invertible linear transformation

$$x(t) = L(t)\bar{x}(t) \tag{2.83}$$

where $L(t)$ and $\dot{L}(t)$ are continuous and bounded matrices, known as a *Lyapunov transformation*. It is also assumed that the system (2.1) and (2.2) is uniformly (differentially) observable in the sense of (1.36).

Lemma 2.1 *The system (2.1) and (2.2) is equivalent to the state representation*

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + L^{-1}(t)B(t)u(t) \tag{2.84}$$

$$y(t) = \bar{C}(t)\bar{x}(t) \tag{2.85}$$

where

$$\bar{A}(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\dot{L}(t)$$

$$= \begin{bmatrix} \bar{A}_{00}(t) & \bar{A}_{01} & \cdots & \bar{A}_{0m} \\ \bar{A}_{10}(t) & \bar{A}_{11} & \cdots & \bar{A}_{1m} \\ \vdots & \vdots & & \vdots \\ \bar{A}_{m0}(t) & \bar{A}_{m1} & \cdots & \bar{A}_{mm} \end{bmatrix} \tag{2.86}$$

$$\bar{C}(t) = C(t)L(t) = [\bar{C}_1(t), 0_{m,n-m}], \quad \bar{C}_1(t) > 0 \tag{2.87}$$

The matrix partitions of $\bar{A}(t)$ that are of time-invariant *structure are*

$$A_{ii} = \left[\begin{array}{c|c} 0_{d_i-2} & I_{d_i-2} \\ \hline 0 & 0_{d_i-2} \end{array}\right], \quad i = 1, \ldots, m;$$

$$\bar{A}_{0i} = [e_i \mid 0_{m,d_i-2}], \quad i = 1, \ldots, m;$$

$$\bar{A}_{ij} = 0_{d_i-1,d_i-1}, \quad i \neq j \quad \text{and} \quad i, j = 1, \ldots, m;$$

where d_i, $i = 1, \ldots, m$, is the observability index of the ith *subsystem and satisfies $\sum_{i=1}^{m} d_i = n$. The matrix partitions A_{i0}, $i = 0, 1, \ldots, m$, contain all the* time-varying *parameters.*

The above companion form is similar to the Luenberger observable companion form and is, in fact, a generalization of Tuel's observable companion form [T9], [Y6] to time-varying linear systems. A procedure for constructing the Lyapunov transformation matrix $L(t)$, in a manner similar to that of (2.72)–(2.74) extended to time-varying systems, is provided by the following algorithm.

Algorithm 2.1 (a) At any time $t \geqslant t_0$ it is possible to uniquely define an $n \times n$ non-singular matrix by eliminating from top to bottom those row vectors of $M(t)$ in (1.36) which are linearly dependent on the preceding rows. It is assumed that the rows of any such non-singular matrix are *index-invariant*, that is to say, if $q_{ij}(t)$ represents the jth row of $Q_i(t)$ at a particular time t, it remains so for all $t \geqslant t_0$. Each integer d_i, $i = 1, \ldots, m$, the observability index of the ith subsystem, is defined as the highest integer for which $q_{d_i i}$ is among the rows of the non-singular matrix.

Construct the non-singular matrix $\hat{M}(t)$ by ordering the selected row vectors q_{ij} according to

$$\hat{M}(t) = \left[\begin{array}{c} \hat{M}_1(t) \\ \hline \hat{M}_2(t) \end{array}\right] \tag{2.88}$$

where

$$\hat{M}_1(t) = \begin{bmatrix} q_{d_1 1} \\ q_{d_2 2} \\ q_{d_m m} \end{bmatrix}$$

and $\hat{M}_2(t)$ consists of q_{ij}, $j = 1, \ldots, m$; $i = 1, \ldots, d_j - 1$ in any order.

(b) Let $h_i(t)$, $i = 1, 2, \ldots, m$, be the ith column of $\hat{M}^{-1}(t)$ where $\hat{M}(t)$ is given by (2.88).

(c) Define vectors $l_{ij} \in R^n$, $i = 1, \ldots, m$; $j = 1, \ldots, d_i$ according to

$$l_{i1}(t) = h_i(t), \quad i = 1, \ldots, m \tag{2.89}$$

and

$$l_{ij}(t) = A(t)l_{ij-1}(t) - \dot{l}_{ij-1}(t)$$
$$i = 1, \ldots, m; j = 2, \ldots, d_i. \tag{2.90}$$

(d) Construct

$$L_0(t) = [l_{1d_1}, l_{2d_2}, \ldots, l_{md_m}]$$

and

$$L_i(t) = [l_{iq_i-1}, l_{iq_i-2}, \ldots, l_{i1}], \quad i = 1, \ldots, m. \tag{2.91}$$

Construct the $n \times n$ matrix $L(t)$ as

$$L(t) = [L_0(t), L_1(t), \ldots, L_m(t)]. \tag{2.92}$$

Given that the system (2.1) and (2.2) has been transformed to its equivalent observable companion form (2.84)–(2.87), we have the following procedure for constructing the minimal-order observer.

(a) Choose the $n - m$ desired observer eigenvalues λ_i, $i = 1, \ldots, n - m$, subject to the two conditions that they have negative real parts for error stability and any complex eigenvalues only occur in complex conjugate pairs for physical realizability.

(b) Determine α_i, $i = 0, 1, \ldots, n - m - 1$ from

$$\prod_{i=1}^{n-m} (\lambda - \lambda_i) = \prod_{i=0}^{n-m-1} \alpha_i \lambda^i + \lambda^{n-m} \tag{2.93}$$

and let

$$\phi = (\alpha_{n-m-1}, \ldots, \alpha_1, \alpha_0)'. \tag{2.94}$$

(c) Let

$$r_i = \sum_{j=1}^{i} (d_j - 1), \quad i = 1, \ldots, n - m - 1 \tag{2.95}$$

and determine the time-invariant $(n - m) \times m$ observer gain matrix V_2 according to

$$V_2 = [-\phi, e_{r_1}, e_{r_2}, \ldots, e_{r_{n-m-1}}]. \tag{2.96}$$

Then, by the relations (2.20) and (2.25) noting that changes of sign are immaterial and P_2 may be assumed equal to the identity matrix, an observer of order $n - m$ may be characterized by

$$T = [V_2, I_{n-m}] \tag{2.97}$$

$$P = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}, \quad V(t) = \begin{bmatrix} I_m \\ -V_2 \end{bmatrix} \bar{C}_1^{-1}(t) \tag{2.98}$$

where $\bar{C}_1(t)$ is as in (2.87). Furthermore, the observer coefficient matrices are

$$D = T\bar{A}(t)P = [-\phi, e_1, e_2, \ldots, e_{n-m-1}] \tag{2.99}$$

$$E(t) = T\bar{A}(t)V(t) \tag{2.100}$$

$$G(t) = T\bar{B}(t). \tag{2.101}$$

Notice that the observer coefficient matrix D is in a *time-invariant phase-variable form.* In particular, the characteristic polynomial of D is identical to (2.93) and the eigenvalues are precisely the *a priori* specified desired eigenvalues of step (a). For the special case of single-output systems ($m = 1$), the minimal-order observer is a time-varying version of (2.68) and (2.69).

2.7 NOTES AND REFERENCES

Both the facility and the advantages of transforming a linear system into a structurally simpler form have been noted by many authors. In the context of observers, the transformation of Section 2.2 is often attributed to Dellon and Sarachik [D5] and Gopinath [G4]. It was also independently arrived at by Cumming [C3] and Newmann [N6] while it seems to have passed unrecognized that the transformation was first used by Bryson and Johansen [B21] in the related problem of linear filtering for coloured noise, discussed in Chapter 6.

The idea of specifying the minimal-order state observer by a single explicit gain matrix is due to Dellon and Sarachik [D5], and follows directly from the transformation of the system into a simpler form. Similar results are obtained by Gopinath [G4], Cumming [C3] and Newmann [N6] for linear time-invariant systems. More recently, an equivalent observer construction, based on the generalized matrix inverse, is presented by Das and Ghoshal [D1]. The time-varying exposition of Section 2.3 is based partly on Yüksel and Bongiorno [Y6] and partly on O'Reilly and Newmann [O25]. In Section 2.4, the parametric observer design for continuous-time linear time-varying systems is based on Johnson [J5] and O'Reilly and Newmann [O25]. For linear time-invariant systems, the design problem reduces to one of conventional eigenvalue assignment for which there are several methods currently available [W10], [P14]. The simplest one, embodied in Theorem 2.4, is due to Gopinath [G4] and effectively reduces the multi-output eigenvalue assignment problem to the simpler well-known single-output problem through restricting the observer gain matrix to be of unity rank. Lemma 2.1 or the result that observability of the pair (A, C) implies that of (A_{22}, A_{12}) is also due to Gopinath [G4].

The multivariable observable canonical form of Section 2.5 was introduced by Luenberger [L11], [L12] and has since enjoyed wide application in observer design, system identification and realization theory because of the structural simplification it affords; see also Brunovsky [B19]. In conventional control studies terminology, the term "canonical form" is often used fairly loosely. Many of these so-called canonical forms are *not* canonical in the strict mathematical sense; for a precise definition of canonical form see Wang and Davison [W2]. The treatment in Section 2.5 follows the original of Luenberger [L11] and has also benefited considerably from the study of Wolovich [W10]. It forms the basis of a computer-aided procedure for the systematic design of the minimal-order state observer by Munro [M21]. See Wolovich [W8] for an extension of the Luenberger companion form and allied minimal-order state observer design to time-varying systems. An equivalent treatment is adopted in Section 2.6 and is based on the use of a time-varying generalization of the Tuel companion form by Yüksel and Bongiorno [Y6].

Chapter 3

Linear State Function Observers

3.1 INTRODUCTION

In Chapters 1 and 2, attention was focussed on the reconstruction of the complete state vector of a linear system. A central result was that, regardless of actual design method, an observer of order $n - m$ with arbitrary stable dynamics fulfils the task. Frequently, however, only some linear function of the system state, typically a linear feedback control law of the form Kx is required to be estimated. It is reasonable to expect that the reconstruction of a linear function of the state vector can be accomplished by an observer of further reduced dimension. Indeed, this is the case and the present chapter is concerned with the delineation of such a class of observer in its two main aspects; namely the minimal order and the stabilization of the observer.

Our investigation begins with the problem of reconstructing a single linear state functional of a multi-output system. Ample use is made of a multivariable system companion form, encountered in Section 2.5, to demonstrate that a suitable low-order observer with arbitrary dynamics exists. It then remains to determine, in Section 3.3, a simple effective design procedure for the arbitrary assignment of the observer eigenvalues (response).

The corresponding multiple functional minimal-order stable observer design problem can be formulated in several different ways. In the first method, (Section 3.4) the problem is recast as a minimal partial stable realization problem or the problem of constructing a minimal-order stable state–space model from input–output data. The second approach (Section 3.5) is to transform the observer constraint equations to a form suitable for the application of decision methods for the unknown design parameters. A common feature of all these methods is the transformation of the system and (or) observer to a co-ordinate basis whereby the number of observer design parameters is substantially reduced.

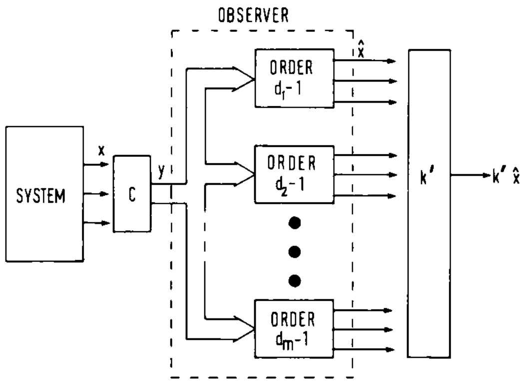

Fig. 3.1 Observing a single linear functional.

3.2 OBSERVING A SINGLE LINEAR STATE FUNCTIONAL

The state reconstruction scheme of Section 2.5 consisted of a composite observer formed from the m subobservers for the m single-output subsystems identified by the multivariable companion form (2.77) and (2.78). In order to reconstruct a single linear state functional, say $k'x$, the same linear state functional of the composite observer output is taken. The scheme is depicted in Fig. 3.1.

The largest block of the observer has $\nu - 1$ eigenvalues that may be chosen arbitrarily and the eigenvalues of the other blocks can be chosen to be a subset of these $\nu - 1$ eigenvalues. In other words, corresponding to each system output y_i there is a transfer function of the form $D_i(s)/D(s)$ from y_i through the observer to $k'\hat{x}$. The polynomial $D(s)$ is the characteristic polynomial of the largest block in the observer, and $D_i(s)$ is a polynomial of degree no greater than that of $D(s)$. It is thereby clear that the observer dynamics need only be of order $\nu - 1$ and the following theorem is established.

Theorem 3.1 *A single linear functional of the state of a linear system can be reconstructed by an observer with $\nu - 1$ eigenvalues that may be chosen arbitrarily (ν is the observability index of the system).*

Noting that $\sum_{i=1}^{m} (d_i - 1) = n - m$ and $\max_i \{d_i\} = \nu$ imply that $\nu - 1 \leqslant n - m$, it is often the case that the order $\nu - 1$ of a linear functional observer is considerably less than the order $n - m$ of a state observer. As a consequence, a linear functional observer often enjoys a substantial reduction in complexity over that afforded by a state observer.

3.3 A GENERAL LINEAR STATE FUNCTIONAL RECONSTRUCTION PROBLEM

Consider the linear time-invariant system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^r$. Assume without loss of generality that the system is completely controllable, completely observable, and that B and C are of full rank.

It is desired to reconstruct several linear functions of the state Kx*, typically a linear feedback control low of the form

$$u = Kx \tag{3.3}$$

where K is an $m \times n$ matrix. For special case $m = 1$, this problem is the same as the single linear state functional reconstruction problem discussed in Section 3.2.

In order to achieve this state function reconstruction an observer of the form

$$\dot{z}(t) = Dz(t) + Ey(t) + Gu(t) \tag{3.4}$$

$$w(t) = Pz(t) + Vy(t) \tag{3.5}$$

will be designed, where $z \in R^p$, and $w \in R^m$ approximates Kx. The observer system (3.4) and (3.5) itself is assumed without loss of generality to be completely observable in that any unobservable states can always be eliminated through defining a lower dimensional observer state vector. As we shall see, the order p of the observer (3.4) and (3.5), as yet unspecified, is less than or equal to that of the minimal-order state observer of Chapter 1 and Chapter 2 ($p \leqslant n - m$).

Definition 3.1 The output $w(t)$ of (3.5) is said to estimate $Kx(t)$ in an asymptotic sense if

$$\lim_{t \to \infty} [w(t) - Kx(t)] = 0. \tag{3.6}$$

In a similar vein to the state observer results of Section 1.7, it is reasonable to suppose that if w estimates Kx, then z estimates some other linear function of x, Tx for instance, and relations similar to (1.76)–(1.78) obtain. In fact, we have the following theorem.

* Unobservable states of (3.1) and (3.2) can be allowed provided they are in the null space of K.

Theorem 3.2 *The completely observable* pth *order observer* (3.4) *and* (3.5) *will estimate* Kx *in the sense of Definition 3.1 if and only if the following conditions hold:*

$$\begin{aligned} &(i)\ D \text{ is a stability matrix} && (3.7)\\ &(ii)\ TA - DT = EC && (3.8)\\ &(iii)\ G = TB && (3.9)\\ &(iv)\ K = PT + VC. && (3.10)\end{aligned}$$

Proof Sufficiency of conditions (i)–(iv) is proved in a manner identical to that of (1.76)–(1.78). Necessity of conditions (i) and (iii) is also proved in analogous fashion. Consider now the proof of conditions (ii) and (iv) where we suppose $\lim_{t\to\infty}(w - Kx) = 0$.

By taking Laplace transforms of (3.1) and (3.3), and (3.4) and (3.5) respectively, this equality is equivalent to the transfer function equality

$$K(sI - A)^{-1}B = [P(sI - D)^{-1}T'(sI - A) + [VC + P(sI - D)^{-1}EC]](sI - A)^{-1}B$$

where the right-hand side expression is the transfer function of the observer (3.4) and (3.5) between input u and output w.

Performing a partial fraction expansion using the identity

$$(sI - X)^{-1} = (s^{-1} + Xs^{-2} + X^2s^{-3} + \cdots)$$

and equating coefficients of s^{-i} to zero yields

$$\begin{bmatrix} T'[P', D'P', (D^2)'P', \ldots, (D^{n-1})'P'] \\ + (C'E' - A'T')[0P'(D')P', \ldots, (D^{n-2})'P'] \\ + [(C'V' - K')00, \ldots, 0] \end{bmatrix} \begin{bmatrix} A^{n-1}B \\ \vdots \\ A \quad B \\ B \end{bmatrix} = 0. \quad (3.11)$$

Now, defining controllability and observability matrices

$$\begin{aligned} \bar{Q} &= [B'(A^{n-1})', \ldots, B'A', B'] \\ Q &= [P', D'P', \ldots, (D^{p-1})'P']. \end{aligned} \quad (3.12)$$

Equation (3.11) is equivalent to the conditions $(C'V' - K' + T'P')\bar{Q} = 0$ and $(T'D' + C'E' - A'T')[Q' : X]\bar{Q}' = 0$ for some matrix X. Thus with (A, B) completely controllable and (D, P) completely observable, $\bar{Q}$ and $[Q' : X]$ are of full rank and consequently these equations establish the necessity of conditions (ii) and (iv), completing the proof. Q.E.D. □

The linear functional observer of Theorem 3.2 is similar in many respects to the state observer of Section 1.7, obtained by setting $K = I$. In particular, conditions (ii) and (iii) and (3.4) imply that the observer error $\varepsilon(t) \triangleq z(t)$

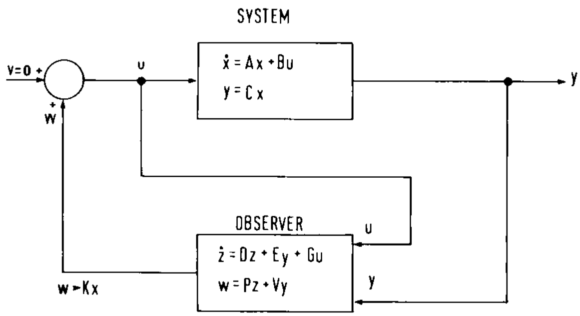

Fig. 3.2 Feedback control law Kx implemented by an observer.

$- Tx(t)$ satisfies

$$\dot{\varepsilon}(t) = D\varepsilon(t) \tag{3.13}$$

and, given condition (i),

$$\lim_{t \to \infty} \varepsilon(t) = 0.$$

Also, if Kx constitutes a linear feedback control law approximated by setting u equal to w, it is readily deduced that the $n + p$ poles of the closed-loop system shown in Fig. 3.2 are given by the eigenvalues of $A + BK$ and the eigenvalues of D. This observation follows from the fact that (3.2), (3.5) and (3.10) imply that

$$w - Kx = P(z - Tx)$$

which together with (3.1) and (3.10) yields the block triangular composite system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A + BK & BP \\ 0 & D \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix}. \tag{3.14}$$

Returning to the observer equation (3.5), the direct feedthrough term V may be constrained to zero in order to prevent additive measurement noise from passing unfiltered into the estimate w of Kx. Such an observer is often referred to as an *observer of Kalman-type*. The usual observer where V is unconstrained is likewise referred to as an *observer of Luenberger-type*.

If the system matrix A and the observer matrix D do not have common eigenvalues, Equation (3.8) possesses a unique solution for T (Luenberger [L10], Gantmacher [G1]). It is, however, unnecessary for the development to make this restrictive assumption.

The observability conditions on the pair (D, P) can be expressed as $\det [Q'Q] \neq 0$ where Q is the observability matrix of (3.12). Actually if the

conditions (3.7)–(3.10) are tested for observers of increasing order $0, 1, 2, \ldots,$ the observability condition need not be tested. To see this, assume that the necessary and sufficient conditions are not satisfied for observers of orders $0, 1, 2, \ldots, p - 1$. Also assume that the sufficient conditions (3.7)–(3.10) are tested for observers of order p, but $\det [Q'Q] = 0$. Then, an observer of order p exists which is not completely observable and as a consequence, a reduced-order observer may be obtained by removing the unobservable states. This contradicts the first assumption and the desired result is established.

With the results of Theorem 3.2 in mind we are now in a position to formally state the general linear state functional reconstruction problem.

The minimal-order observer design problem:
Determine an observer of minimal state dimension p and matrices D, E, G, P and V such that conditions (i)–(iv) are satisfied. A satisfactory solution of the problem hinges on finding matrices D, E, G, P and V of simple structure that will satisfy conditions (i)–(iv) for an integer p of minimum value.

3.3.1 Observer design for a single linear state functional

Before mounting an assault on the general multiple linear state functional observer design problem, let us pause to consider again that of the single linear state functional case. The importance of the single state functional case in its own right stems largely from the fact that Theorem 3.1 ensures that a suitable observer of order $\nu - 1$ exists with eigenvalues that may be completely arbitrarily assigned. Although the derivation culminating in Theorem 3.1 also provides a design procedure for reconstructing $k'x$, a rather simpler method that is both computationally attractive and readily applied to large-scale systems may now be considered in the light of Theorem 3.2.

Specifically, by (3.7) the (stable) observer dynamics are determined by D, which may be chosen arbitrarily, with the mild restriction that its eigenvalues $(\lambda_1, \ldots, \lambda_{\nu-1})$ are distinct and different from the eigenvalues of A. This latter assumption guarantees that (3.8) has a unique solution [L10], [G1]. Also, in the present single state functional case Equation (3.10) reduces to

$$k' = p'T + v'C. \tag{3.15}$$

The design of a $(\nu - 1)$th order observer with arbitrary (stable) eigenvalues for the asymptotic state reconstruction of the single linear state functional $k'x$ is summarized in the following theorem.

Theorem 3.3 *Let U be a matrix of self-conjugate column eigenvectors of D so that*

$$D = U\Delta U^{-1} \tag{3.16}$$

where $\Delta = \operatorname{diag}(\lambda_1, \ldots, \lambda_{v-1})$. *Let*

$$p' = \alpha' U^{-1} \tag{3.17}$$

where α' is the $(v-1)$-dimensional sum vector $[1, 1, \ldots, 1]$. *Then*

$$E = UM \tag{3.18}$$

where v' and the rows of M, $(m'_1, \ldots, m'_{v-1})$, are obtained as the solution of the set of linear equations (3.19).

$$[v', m'_1, m'_2, \ldots, m'_{v-1}] \begin{bmatrix} C \prod_{j=1}^{v-1} (A - \lambda_j I) \\ C \prod_{j=2}^{v-1} (A - \lambda_j I) \\ C \prod_{\substack{j=1 \\ j \neq 2}}^{v-1} (A - \lambda_j I) \\ \vdots \\ C \prod_{j=1}^{v-2} (A - \lambda_j I) \end{bmatrix} = k' \prod_{j=1}^{v-1} (A - \lambda_j I) \tag{3.19}$$

$$G = URB \tag{3.20}$$

where R is a $(v-1) \times n$ matrix with rows $(r'_1, r'_2, \ldots, r'_{v-1})$ where

$$r'_j = m'_j C(A - \lambda_j I)^{-1}, \quad j = 1, \ldots, v-1. \tag{3.21}$$

Equations (3.19) are consistent, and the solution is unique if $mv = n$, and non-unique if $mv > n$.

Proof Setting $U^{-1}T = R$ and $U^{-1}E = M$, (3.16) and (3.8) give

$$RA - \Delta R = MC \tag{3.22}$$

from which, since Δ is diagonal, we obtain (3.21). Equations (3.15) and (3.17) yield

$$v'C + \alpha' R = k'. \tag{3.23}$$

Substitute (3.21) into (3.23) to obtain

$$v'C + \sum_{j=1}^{v-1} m'_j C(A - \lambda_j I)^{-1} = k'. \tag{3.24}$$

Postmultiplication of (3.24) by $\prod_{j=1}^{v-1}(A-\lambda_j I)$ and rearranging give (3.19). Equation (3.18) follows from the definition of M while (3.20) follows from (3.9) and the definition of R above.

The only unknowns in (3.19) are the mv components of the vector $[v', m'_1, \ldots, m'_{v-1}]$. It can be verified by elementary row operations on the left-hand side of (3.19) that, provided the λ_j are distinct, this matrix has the same rank as that of (1.33), namely, rank n, by definition of complete observability. Hence, Equations (3.19) are consistent and possess a unique (non-unique) solution if $mv = n$ $(mv > n)$. Q.E.D. □

3.4 MINIMAL-ORDER OBSERVER DESIGN VIA REALIZATION THEORY

In order to formulate the general multiple functional observer design problem as a minimal stable realization problem, it is first convenient to simplify the constraint Equations (3.8)–(3.10) by exhibiting more of the structure of the system in (3.1)–(3.3) in special co-ordinates. Indeed, we recall from Section 2.2 that the system (3.1)–(3.3) may be assumed without loss of generality to be described by

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \tag{3.25}$$

$$y(t) = Cx(t) = [0 \quad I_r]x(t) \tag{3.26}$$

$$u(t) = Kx(t) = [K_1 \quad K_2]x(t) \tag{3.27}$$

where K_1 is $m \times (n-r)$ and K_2 is $m \times r$. If T is conformally partitioned as

$$T = [T_1 \quad T_2]$$

then, substitution of A, C, K and T into (3.8) and (3.10) yields the equations

$$0 = T_1 A_{11} + T_2 A_{21} - DT_1 \tag{3.28}$$

$$E = T_1 A_{12} + T_2 A_{22} - DT_2 \tag{3.29}$$

$$K_1 = PT_1 \tag{3.30}$$

$$V = K_2 - PT_2. \tag{3.31}$$

It is observed that if D, P, T_1 and T_2 are known then E and V are easily obtained from (3.29) and (3.31) respectively. Thus, the problem reduces to obtaining matrices D, P, T_1 and T_2 that satisfy (3.28) and (3.30) such that D is a stability matrix and p is a minimum.

A further simplification is obtained by considering the system (3.25)–(3.27)

in a second basis with (A_{11}, A_{21}) in the Luenberger observable form (see Section 2.5).

$$
\begin{aligned}
A_{11} &= [e_2 e_3 \cdots e_{d_1} - \alpha_1 e_{d_2+2} \cdots e_{d_2} - \alpha_2 \cdots e_{d_{r-1}+2} \cdots e_{n-r} - \alpha_r] \\
A_{21} &= [00 \cdots 0e_1 0 \cdots 0e_2 0 \cdots 0e_r] \qquad (3.32) \\
d_i &= \sum_{j=1}^{i} (d_j - 1)
\end{aligned}
$$

where $d_i \in R^{n-r}$ is a non-zero vector, the set of integers $\{d_i\}$ is the set of observability indices of the pair (A, C), and $v = \max\{d_i\}$ is the observability index of (A, C). Recall from Lemma 2.1 that this is legitimate since if (A, C) is an observable pair, then (A_{11}, A_{21}) is also an observable pair with observability indices one less than the corresponding observability indices of (A, C). For notational simplicity, it is assumed in (3.32) that every one of the d_i is greater than or equal to 2.

Now label the columns of T_1 and T_2 as follows:

$$
\begin{aligned}
T_1 &= [t_{11} t_{12} \cdots t_{1d_1-1} t_{21} \cdots t_{2d_2-1} \cdots t_{r1} \cdots t_{rd_r-1}] \\
T_2 &= [\tilde{t}_1 \tilde{t}_2 \cdots \tilde{t}_r].
\end{aligned} \qquad (3.33)
$$

Substituting (3.32) and (3.33) into (3.28) gives the following relations for the columns of T_1 and T_2:

$$
\begin{aligned}
t_{ij} &= D^{j-1} t_{i1} \quad \begin{array}{l} i = 1, 2, \ldots, r \\ j = 2, 3, \ldots, d_i - 1 \end{array} \\
\tilde{t}_i &= D^{d_i-1} t_{i1} + T_1 \alpha_1 \quad i = 1, 2, \ldots, r
\end{aligned} \qquad (3.34)
$$

where α_i is defined in (3.32). For notational simplicity, let

$$t_{i1} = t_i \quad i = 1, 2, \ldots, r$$

in the sequel. The set of vectors $t_i, Dt_i, \ldots, D^{d_i-2} t_i$ is referred to as the ith chain of T_1.

Thus the problem has reduced to obtaining $D, t_1, \ldots, t_r$, and P such that (3.30) is satisfied, D is a stability matrix, and p is as small as possible.

Analogous to T_1 in (3.33) label the columns of K_1 as

$$K_1 = [k_{11} k_{12} \cdots k_{1d_1-1} k_{21} \cdots k_{2d_2-1} \cdots k_{r1} \cdots k_{rd_r-1}].$$

The vectors $k_{i1}, k_{i2}, \ldots, k_{id_i-1}$ are referred to as the ith chain of K_1, and the correspondence between the columns of T_1 and K_1 is given by

$$k_{ij} = PD^{j-1} t_i \quad \begin{array}{l} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, d_i - 1 \end{array}. \qquad (3.35)$$

Let $N = \nu - 1$. In the most general case, all the chains of K_1 and of T_1 have different lengths. The lengths of all chains can be made equal, and an augmented form of (3.30) can be defined as

$$\hat{K}_1 = P\hat{T}_1 \tag{3.36}$$

by adding $\nu - d_i$ (as yet unspecified) vectors $\xi_{id_i}, \xi_{id_i+1}, \ldots, \xi_{iN}$ that satisfy

$$\xi_{ij} = PD^{j-1}t_i \quad \begin{matrix} i = 1, 2, \ldots, r \\ j = d_i, \ldots, N \end{matrix}$$

to the end of every chain of K_1, and adding the vectors $D^{d_i-1}t_i, \ldots, D^{N-1}t_i$ to the end of the corresponding chain of T_1. That is, the columns of $\hat{K}_1$ are given by

$$\hat{k}_{ij} = \begin{cases} k_{ij} & \text{for } 1 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant d_i - 1 \\ \xi_{ij} & \text{for } 1 \leqslant i \leqslant r \text{ and } d_i \leqslant j \leqslant N \end{cases}.$$

From the columns of $\hat{K}_1$, define a set of $m \times r$ matrices as

$$L_i = [l_{1i}l_{2i} \cdots l_{ri}] = [\hat{k}_{1i}\hat{k}_{2i} \cdots \hat{k}_{ri}] \quad i = 1, 2, \ldots, N$$

and further define a $p \times r$ matrix Q as

$$Q = [t_1 t_2 \cdots t_r].$$

Then, an equivalent form of (3.36) is obtained by rearranging the columns of $\hat{K}_1$ and $\hat{T}_1$ as follows.

$$[L_1 L_2 \cdots L_N] = P[QDQ \cdots D^{N-1}Q].$$

These results are summarized in the following lemma.

Lemma 3.1 *The observer (3.4) and (3.5) with E, G, V, and T given by (3.29), (3.9), (3.31), and (3.33), (3.34), respectively, is a minimal-order observer that estimates Kx in the sense of Definition 3.1, if and only if the following conditions are satisfied:*

(i) $L_i = PD^{i-1}Q \quad i = 1, 2, \ldots, N$ (3.37)

(ii) p is a minimum

(iii) D is a stability matrix.

Conditions (i) and (ii) of Lemma 3.1 define a minimal partial realization problem in the context of which D, Q and P are regarded as system, input and output matrices. That is, a state–space realization (D, Q, P) is minimal in the sense that it has the lowest order or state dimension among all realizations having the same transfer function $P(sI - D)^{-1}Q$. Any (non-unique) minimal-order realization may be unstable, and so it is important to consider higher order realizations that satisfy conditions (i)–(iii) which define a minimal partial

stable realization problem. The dimension of the minimal partial realization problem solution serves as a lower bound on the dimension of the minimal-order stable observer (stable realization).

3.4.1 The minimum partial realization

It is well known [K13], [P10] that, as the dual of observability, given a completely controllable pair (D, Q), the following r linear dependence equations can be defined:

$$D^{\mu_i}t_i = -\sum_{j=1}^{r}\sum_{k=1}^{\sigma_{ij}} \gamma_{ijk}D^{k-1}t_j, \quad i = 1, 2, \ldots, r \tag{3.38}$$

where

$$\sigma_{ij} = \begin{cases} \mu_i + 1 & \text{for } \mu_i < \mu_j, i > j \\ \mu_i & \text{for } \mu_i \leqslant \mu_j, i \leqslant j, \\ \mu_i & \text{for } \mu_i \geqslant \mu_j, \forall i,j \end{cases}$$

$\{\gamma_{ijk}\}$ are real scalars, and $\{\mu_i\}$ are the controllability indices of the pair (D, Q). The $\{\mu_i\}$ satisfy the relations $p = \mu_1 + \mu_2 + \cdots + \mu_r$ and $\mu = \max\{\mu_i\}$ where μ is the controllability index of (D, Q).

Based on Equation (3.38) part of the realization algorithm consists of identifying the parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ from the given data sequence $\{L_1, L_2, \ldots, L_N\}$ as follows. Once the $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ are found, the triple (D, Q, P) is uniquely specified (*modulo* a basis change) and may be realized with the pair (D, Q) in Luenberger's first companion form.

Associated with this sequence is the truncated generalized $mN \times rN$ Hankel matrix [K18].

$$\mathscr{H}_{N,N} = \begin{bmatrix} L_1 & L_2 & & \cdots & L_{N-1} & L_N \\ L_2 & L_3 & & \cdots & L_N & * \\ \vdots & \vdots & & & \vdots & \vdots \\ L_{N-1} & L_N & * & \cdots & * & * \\ L_N & * & * & \cdots & * & * \end{bmatrix}$$

where the asterisks (*) represent the elements of an unspecified extension sequence $\{L_{N+1}, \ldots, L_{2N-1}\}$. The extension sequence is chosen so that a column of $\mathscr{H}_{N,N}$ which is linearly dependent upon its preceding columns before the selection of the extension sequence remains thus linearly dependent after selection of the extension sequence. Now after examining the columns of $\mathscr{H}_{N,N}$ for linear dependence, the first column found to be linearly dependent upon its preceding columns has the form

$$\begin{bmatrix} l_{i,\mu_i+1} \\ l_{i,\mu_i+2} \\ \vdots \\ l_{i,N} \end{bmatrix} = \begin{bmatrix} PD^{\mu_i}t_i \\ PD^{\mu_i+1}t_i \\ PD^{N-1}t_i \end{bmatrix}$$

for some integer $1 \leqslant i \leqslant r$ (for notational simplicity, we assume $\mu_i > 0$). This identifies the ith controllability index of (D, Q). The solution to the following $m(N - \mu_i)$ equations gives the $\sigma_i = \sigma_{i1} + \sigma_{i2} + \cdots + \sigma_{ir}$ unknowns $\{\gamma_{ijk}\}$:

$$\begin{bmatrix} l_{i,\mu_i+1} \\ l_{i,\mu_i+2} \\ \vdots \\ l_{i,N} \end{bmatrix} = - \begin{bmatrix} l_{1,1} & \cdots & l_{i-1,\mu_i+1} \\ l_{1,2} & \cdots & l_{i-1,\mu_i+2} \\ \vdots & & \vdots \\ l_{1,N-\mu_i} & \cdots & l_{i-1,N} \end{bmatrix} \begin{bmatrix} \gamma_{i11} \\ \gamma_{i21} \\ \vdots \\ \gamma_{ii-1\mu_i} \end{bmatrix}.$$

This equation can be rewritten compactly as

$$h_i = -\mathscr{M}_i \gamma_i \tag{3.39}$$

The remaining columns of $\mathscr{H}_{N,N}$ are examined for linear dependence, skipping over the columns associated with t_i vectors for which a controllability index has already been obtained. The next column found to be linearly dependent upon its preceding columns gives us another controllability index and its associated $\{\gamma_{ijk}\}$ parameters, and so on.

Once the $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ are found, the triple (D, Q, P) is uniquely specified (*modulo* a basis change) and may be realized in the following co-ordinates with the pair (D, Q) in Luenberger's first companion form [L12].

$$D = [D_{ij}], \quad i, j = 1, 2, \ldots, r$$

$$D_{ii} = [e_2 e_3 \cdots e_{\mu i} - \gamma_{ii}], \quad D_{ij} = [0 \cdots 0 - \gamma_{ji}] \tag{3.40}$$

$$\gamma'_{ii} = [\gamma_{ii1} \cdots \gamma_{ii\mu_i}], \quad \gamma'_{ji} = [\gamma_{ji1} \cdots \gamma_{ji\sigma_{ji}} 0 \cdots 0]$$

$$Q = [t_1 t_2 \cdots t_r] \tag{3.41}$$

$$t_1 = e_1, \quad t_2 = e_{\mu_1+1}, \quad \ldots, \quad t_r = e_{\mu_1+\cdots+\mu_{r-1}+1}$$

$$P = [p_{11} \cdots p_{1\mu_1} p_{21} \cdots p_{2\mu_2} \cdots p_{r1} \cdots p_{r\mu_r}]$$

$$p_{ij} = l_{ij} = \hat{k}_{ij} \quad \begin{array}{l} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, \mu_i \end{array} \tag{3.42}$$

where D_{ij} is a $\mu_i \times \mu_j$ matrix and $\hat{k}_{ij}$ is the jth vector in the ith chain of $\hat{K}_1$.

These results are summarized in the following theorem.

Theorem 3.4 *Given the sets of parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ such that a triple (D, Q, P)*

(i) satisfies (3.37)

(ii) has D a stability matrix, and

(iii) has p as small as possible,

then an observer designed using (3.9), (3.29), (3.31) and (3.40)–(3.42) is a minimal-order observer for Kx in the sense of Definition 3.1.

The minimal dimension of the observer from (3.40) is $p = \sum_{i=1}^{r} \mu_i$. If this realization is unstable, higher order realizations are obtained by systematically increasing the size of the minimal controllability indices $\{\mu_i\}$. This procedure increases the number of degrees of freedom in the characteristic polynomial of D and is terminated when sufficient freedom to stabilize D is obtained. The values of $\{\mu_i\}$ are bounded by

$$\mu_{iM} \leqslant \mu_i \leqslant d_i - 1 \quad i = 1, 2, \ldots, r.$$

The lower bound is the μ_{iM} associated with the order p_M of the minimal, not necessarily stable, partial realization. The upper bound is due to the result of Section 3.2 that sub-observers of order $d_i - 1$ and arbitrary poles (eigenvalues) can always be designed for a subsystem of (3.25) of order d_i.

It should be pointed out that the realization problem (3.37) may be solved from the dual point of view to give the dual of (3.40)–(3.42) in observable companion form. The observability indices obtained by the dual realization procedure are the observability indices of the observer or of the pair (D, P). Quite often the dual formulation is preferable because it may be simpler to formulate and (or) may have computational advantages.

3.4.2 Arbitrary observer eigenvalues

Hitherto our main concern was that any observer designed by the realization procedure be stable. It is often additionally desired that an observer attain a prescribed degree of stability (good transient response) through arbitrary eigenvalue assignment. As remarked previously, this generally necessitates an observer of higher dimension and it is of interest to determine the minimal-order of an observer with such arbitrary dynamics. Although results in the general problem are difficult to obtain, necessary and sufficient conditions to achieve the minimal order observer with arbitrary eigenvalues can be stated for the cases $m \geqslant 1$, $r = 1$ and $m = 1$, $r \geqslant 1$.

Theorem 3.5 *An observer of state dimension p and arbitrary stable eigenvalues.*

(*i*) *estimates Kx in the sense of Definition 3.1 for a single output system ($r = 1$) if and only if $p \geqslant n - 1$; and*

(*ii*) *estimates $k'x$ ($m = 1$) in the sense of Definition 3.1 for a multiple output system if and only if $p \geqslant d_q - 1$, where $d_q - 1 \leqslant v - 1$ is the length of the longest non-zero chain of k_1'.*

Proof See Roman and Bullock [R8]. □

3.5 MINIMAL-ORDER OBSERVER DESIGN VIA DECISION METHODS

Decision methods [T2], [A7] in the present context refer to the existence of, and the determination of, solutions v of equalities $f(v) = 0$ and inequalities $g(v) < 0$ where f and g are each a set of real multivariable polynomials in v (polynomials in each of the components of v). In the first instance, decision methods determine in a finite number of steps, involving only rational operations, whether or not a vector v exists such that $f(v) = 0$ and $g(v) < 0$. Unfortunately, the number of steps increases exponentially with the number of unknowns (elements of v) and also the number of inequalities. When such a v is known to exist, the methods of Tarski [T2] provide a solution using polynomial factorization. Again, a large number (theoretically infinite) of steps are required for the solution procedure.

To see, however, the applicability of decision methods to the problem at hand, recall from Lemma 3.1 that necessary and sufficient conditions for a minimal-order observer to exist are that the triple (D, Q, P) constitute a minimal realization and that D is a stability matrix. The parameters $\{\mu_i\}$ and $\{\gamma_{ij}\}$ of the canonical form realization (3.38) are determined for a given data sequence $\{L_1, L_2, \ldots, L_N\}$ from the solution of (3.39). Equation (3.39) is a multivariable polynomial equality and may be tested by the above decision methods for a solution.

Turning now to the stability requirement on the observer matrix D, the stability (negative eigenvalues) of D may be interpreted in terms of its characteristic polynomial

$$\beta(s) = \prod_{i=1}^{n/2} [s^2 + s(w_i^2 + \varepsilon) + (v_i^2 + \varepsilon)] \quad \text{for } n \text{ even}$$

$$= (s + w_i^2 + \varepsilon) \prod_{i=1}^{(n-1)/2} [s^2 + s(w_i^2 + \varepsilon) + (v_i^2 + \varepsilon)]$$

for n odd for ε some small positive constant. This is consistent with choosing the observer in a co-ordinate basis where D is the diagonal sum of the blocks.

$$\begin{bmatrix} 0 & 1 \\ -v_1^2 - \varepsilon & -w_i^2 - \varepsilon \end{bmatrix}.$$

Using $\bar{D}$ to denote a vector consisting of v_i and w_i for all i, it is a simple matter to express β_i, the coefficients of $\beta(s)$, as an explicit function of $\bar{D}$. Denoting this function by $\beta_i(\bar{D})$, the Cayley-Hamilton theorem (Appendix A) for the polynomial

$$\beta(s) = \sum_{i=0}^{p} \beta_i(\bar{D}) = \det(sI - D) = 0$$

is given by the equation

$$\beta(D) = \sum_{i=0}^{p} \beta_i(\bar{D})D^i = 0. \tag{3.43}$$

From (3.35) and (3.43) we have that the vector $\bar{D}$ satisfies the multivariable polynomial equality

$$\sum_{q=0}^{p} \beta_q(\bar{D})k(i, j + q) = 0. \tag{3.44}$$

In summary, the realization and stability multivariable polynomial equalities, Equations (3.39) and (3.44) respectively, may be solved using the decision methods of Anderson *et al.* [A7] and Tarski [T2]. That is, decision methods may be applied to (3.39) and (3.44) for systematically increasing values of p until solutions of (3.39) and (3.44) are shown to exist. Then, these methods can be applied to determine suitable parameters $\{\bar{D}_i, \gamma_{ij}, \mu_i\}$ from which the stable minimal-order observer can be constructed ($\bar{D}_i$ corresponds to the ith diagonal block of the observer matrix, D_{ii}, $i = 1, \ldots, r$).

3.6 NOTES AND REFERENCES

The possibility that an observer of considerably reduced dimension than that of a state observer might be employed to reconstruct a single linear state functional was first explored by Bass and Gura [B5] and Luenberger [L11]. Section 3.2 follows Luenberger's original lead on exploiting a multi-output observable companion form to derive an observer of order $\nu - 1$ with arbitrary (stable) eigenvalues (ν is the system observability index). The more computationally attractive design procedure of Theorem 3.3 is based on Murdoch [M23]. An additional advantage of this method is its ready extension to further reducing the observer order by accepting constraints on the observer eigenvalues [M25]. Alternative design procedures for observing a

scalar linear function of the state of a multiple-output system are presented in [W18], [M16], [R9] and [J1].

Progress on the general problem of reconstructing vector linear functions of the state has been relatively slow due to the greater complexities of the problem. The first significant contribution, upon which Section 3.3 is partly based, is due to Fortmann and Williamson [F6]. They obtain necessary and sufficient conditions for an observer of minimal order to reconstruct a vector linear function of the state for single-output systems.

The treatment of the general vector state function problem for the multiple output system case in Section 3.4 using minimal partial realization theory and in Section 3.5 using decision methods is based on Roman and Bullock [R8] and Moore and Ledwich [M18] respectively. See also Murdoch [M24] for a simple sequential design procedure which may not, however, yield an observer of minimal order. Another method of Fairman and Gupta [F4] is based in the reduction of a state observer for a system in Luenberger companion form where the state functionals are treated as if they were additional outputs. By contrast, the approach of Sirisena [S15] does not require transformation of the system into Luenberger companion form. Other expositions with further results on minimal order and structure from the frequency-domain point of view [W1], [R10], and using a geometric approach [S4] are the subjects of Chapter 9 and Chapter 10.

Finally, it is worth noting that a corresponding observer theory for linear state function reconstruction may be developed for discrete-time linear time-invariant systems. The problem of minimal-order observer design for the state function reconstruction of discrete-time linear systems in a minimum number of time steps is discussed in Chapter 5.

Chapter 4

Dynamical Observer-Based Controllers

4.1 INTRODUCTION

The idea of feedback is central to control system studies. In broad terms, feedback control may be described as the employment of a control strategy, devised on the basis of measurable system outputs, to the system inputs so as to regulate the system variables to some reference or equilibrium values. The advantages enjoyed by the judicious application of feedback in control systems design are well known; they include stabilization, improved stability margins, tuned transient response, noise rejection and reduced sensitivity to system parameter variations.

In Chapter 3, we discussed the design of low-order observers for the reconstruction of a linear function of the system state, in particular that of a linear state feedback control law. Through operating on the available system measurements, the observer supplements the open-loop system dynamics to asymptotically generate the desired feedback control policy. In this sense, the observer shown in Fig. 3.2 may be regarded as forming part of an overall closed-loop feedback configuration or observer-based compensation scheme for the original open-loop system.

The present chapter explores further the possibilities of linear feedback control for systems with inaccessible state, considered in Chapter 1. In particular, the synthesis of a dynamical controller based alternatively on the minimal-order state observer, described in Chapter 2, is considered. Although generally of higher order than its linear state function observer counterpart, a state observer-based control scheme is of a less complex structure and is consequently easier to design.

Section 4.2 continues the discussion of Section 1.5 on the use of linear state feedback to regulate systems of which the state is completely accessible. It is shown that the design freedom left after eigenvalue assignment is to be able to choose an "allowable" set of eigenvectors. The need to limit the control energy in achieving state regulation, due to actuator constraints or saturation

problems, motivates the second design procedure of linear quadratic control.

Section 4.3 and Section 4.4 extend the linear state feedback policies, devised for systems with completely measurable state, to observable and controllable systems with inaccessible state. In essence, the design procedure is as follows: constrain the form of the controller to be that of a constant feedback gain matrix which operates linearly upon an estimate $\hat{x}$ of the actual inaccessible state x, where $\hat{x}$ is generated by a minimal-order state observer; then choose the feedback matrix and observer gain matrix so as to either achieve eigenvalue–eigenvector assignment (Section 4.3) or linear quadratic optimization (Section 4.4) of the overall system. In either case, the ease of closed-loop system compensation is greatly facilitated by the inherent *separation* of the observer-based controller scheme into the two independent problems of observer design and feedback controller design. Finally, in Section 4.5 it is demonstrated that the transfer–function matrix of a closed-loop system incorporating a state observer is identical to the transfer–function matrix of a closed-loop system employing direct state feedback, were the state available.

4.2 LINEAR STATE FEEDBACK CONTROL

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.1}$$

$$y(t) = Cx(t) \tag{4.2}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^r$ is the control input and $y(t) \in R^m$ is the system output. It is assumed that the linear system (4.1) and (4.2) is completely controllable [stabilizable] and completely observable [detectable]. Suppose that the state $x(t)$ of the open-loop system (4.1) is completely accessible; then we saw in Section 1.5 that the application of the linear feedback control law

$$u(t) = Fx(t) + v(t) \tag{4.3}$$

to the controllable system (4.1) results in the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) + Bv(t) \tag{4.4}$$

of which the eigenvalues of $A + BF$ can be arbitrarily assigned by a suitable choice of the gain matrix F. A schematic of the closed-loop system realization is presented in Fig. 4.1. As to the actual choice of stabilizing feedback gain matrix F, the various design methods for asymptotic state reconstruction and observer eigenvalue assignment described in Section 2.4 and Section 2.5 have direct application to the dual control system stabilization problem and vice versa. Indeed the notion of duality has been raised several times already in

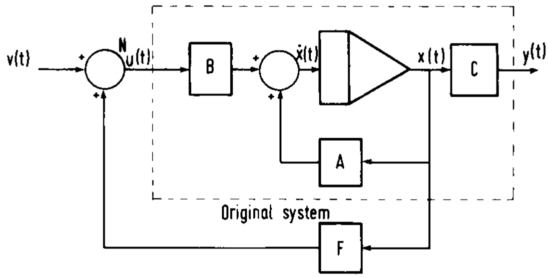

Fig. 4.1 Closed-loop system.

connection with the dual system concepts of observability and controllability. An efficacious attribute of duality is that any property or result associated with the system $\{A, B, C\}$ can usually be established for the dual system $\{A', B', C'\}$ without repeating a proof.

Now while the overall speed of response of the closed-loop system (4.4) is determined by its eigenvalues, the "shape" of the transient response depends to a large extent on the closed-loop eigenvectors. For, unlike the single-input case, specification of the closed-loop eigenvalues does not define a unique closed-loop system. The freedom offered by this non-uniqueness beyond specification of the closed-loop eigenvalues is to be able to choose an "allowable" set of closed-loop eigenvectors. For the case where the closed-loop eigenvalues are distinct, Theorem 4.1 provides necessary and sufficient conditions for a gain matrix F to exist which yields the prescribed eigenvalues and eigenvectors. The proof includes a procedure for computing F. To aid the development, we associate with each number λ the matrix

$$S_\lambda = [\lambda I - A, B] \tag{4.5}$$

and a compatibly partitioned matrix

$$K_\lambda = \begin{bmatrix} N_\lambda \\ M_\lambda \end{bmatrix} \tag{4.6}$$

whose columns constitute a basis for kernal $\{S_\lambda\}$.

Theorem 4.1 *Let λ_i, $i = 1, \ldots, n$, be a set of numbers in which complex numbers occur in complex conjugate pairs. There exists a real matrix F such that*

$$(A + BF)v_i = \lambda_i v_i, \quad i = 1, \ldots, n \tag{4.7}$$

if and only if the following three conditions are satisfied for $i = 1, \ldots, n$.

(*i*) *The vectors v_i are linearly independent in ρ^n.**

* The finite-dimensional vector spaces R^n and ρ^n are defined, respectively, over the field of real numbers and the field of complex numbers.

(*ii*) $v_i = v_j^*$ *whenever* $\lambda_i = \lambda_j^*$ (*complex conjugate of* λ_j).

(*iii*) $v_i \in \text{span } \{N_{\lambda_i}\}$, *the subspace spanned by the columns of* N_{λ_i}. *If F exists and* rank $B = r$, *then F is unique.*

Proof (*sufficiency*) Suppose that v_i, $i = 1, \ldots, n$, are chosen to satisfy conditions (i)–(iii). Since $v_i \in \text{span } \{N_{\lambda_i}\}$ (condition (iii)), then v_i can be expressed as $v_i = N_{\lambda_i} k_i$ for some vector $k_i \in R^r(\rho^r)$, which implies that

$$(\lambda_i I - A)v_i + BM_{\lambda_i} k_i = 0.$$

If F is chosen so that $-M_{\lambda_i} k_i = Fv_i$, then $[\lambda_i I - (A + BF)]v_i = 0$. What remains in the proof is to show that a real matrix F satisfying

$$\begin{aligned} F[v_1 v_2 \cdots v_n] &= [w_1 w_2 \cdots w_n] \\ w_i &= -M_{\lambda_i} k_i \end{aligned} \tag{4.8}$$

can always be constructed.

If all n eigenvalues are real numbers, then v_i, w_i are vectors of real numbers and the matrix $[v_1 v_2 \cdots v_n]$ is non-singular. For this case

$$F = [w_1 w_2 \cdots w_n][v_1 v_2 \cdots v_n]^{-1}. \tag{4.9}$$

For the case where there are complex eigenvalues, assume that $\lambda_1 = \lambda_2^*$. Condition (ii) states that $v_1 = v_2^*$ which implies that $w_1 = w_2^*$. The equation which must be solved then is

$$F[v_{1R} + jv_{1I}, v_{1R} - jv_{1I}, V] = [w_{1R} + jw_{1I}, w_{1R} - jw_{1I}, W] \tag{4.10}$$

where the columns of V and W are v_i, $i = 3, \ldots, n$, and w_i, $i = 3, \ldots, n$, respectively. Multiplication of both sides of (4.22) from the right by the non-singular matrix

$$\left[\begin{array}{cc|c} \frac{1}{2} & -j\frac{1}{2} & 0 \\ \frac{1}{2} & +j\frac{1}{2} & \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right]$$

yields the equivalent equation

$$F[v_{1R}, v_{1I}, V] = [w_{1R}, w_{1I}, W].$$

Since v_i, $i = 1, \ldots, n$, are independent, the columns of $[v_{1R}, v_{1I}, V]$ are linearly independent. The procedure can obviously be applied for all pairs of eigenvalues.

Proof (*necessity*) See [M15]. Q.E.D. □

Our review of linear state feedback control schemes concludes with the statement and solution of the now classical optimal linear regulator problem.

Time-invariant linear quadratic control problem:
Given the linear time-invariant system (4.1), find the control vector $u(t) \in R^r$, $t \in [0, \infty]$, such that the quadratic cost functional

$$J = \int_0^\infty [x'(t)Qx(t) + 2x'(t)Mu(t) + u'(t)Ru(t)]\, dt \tag{4.11}$$

is minimized where the weighting matrices satisfy $Q = Q' \geq 0$, $R = R' > 0$, $Q - MR^{-1}M' \equiv D_1 D_1' \geq 0$ but are otherwise arbitrary.

The solution to this optimization problem is embodied in the following theorem.

Theorem 4.2 *The solution to the linear quadratic optimal control problem is of the linear feedback form*

$$u(t) = Fx(t) \tag{4.12}$$

in which the $r \times n$ constant gain matrix F is given by

$$F = -R^{-1}(B'\hat{K} + M') \tag{4.13}$$

where $\hat{K}$ is an $n \times n$ symmetric solution of the non-linear algebraic Riccati equation

$$0 = A'\hat{K} + \hat{K}A + Q - (\hat{K}B + M)R^{-1}(\hat{K}B + M)'. \tag{4.14}$$

Furthermore, the minimum value of J is $x'(0)Kx(0)$.

Proof We assume that the system (4.1) under the feedback control (4.12) is stable, otherwise the integral J will diverge as $t \to \infty$. Upon expansion of the integrand, with the use of (4.14), the cost functional may be rearranged in the more suggestive form

$$\begin{aligned} J = \int_0^\infty & [\{u + R^{-1}(B'\hat{K} + M')x\}'R\{u + R^{-1}(B'\hat{K} + M')x\} \\ & - (Ax + Bu)'\hat{K}x - x'\hat{K}(Ax + Bu)]\, dt \\ = \int_0^\infty & [\{u + R^{-1}(B'\hat{K} + M')x\}'R\{u + R^{-1}(B'\hat{K} + M')x\} \\ & + x'(0)\hat{K}x(0) - x'(\infty)\hat{K}x(\infty)] \end{aligned} \tag{4.15}$$

using (4.1). If the closed-loop system (4.4) ($v = 0$) is stable, the last term of (4.15) is zero. Thus, J is minimized if and only if $u(t)$ is given by (4.12) and (4.13), whereupon substitution into (4.15) yields the optimal (minimum) cost $x'(0)\hat{K}x(0)$. Q.E.D. □

The linear feedback control law (4.12) of Theorem 4.2, often referred to as the optimal linear regulator, is of the same basic structure as (4.3) treated previously from the point of view of closed-loop eigenvalue (and eigenvector) assignment. One difference, however, is the absence of an external input (i.e., $v(t) = 0$). In the proof of Theorem 4.2 it is recalled that we assumed that the closed-loop system is stable. The existence, uniqueness and stability of the linear regulator solution (4.12) and (4.13) are assured by the conditions of the next lemma.

Lemma 4.1 *If the pair (A, B) is completely controllable [stabilizable] and the pair $(A - BR^{-1}M', D_1)$ is completely observable [detectable], where $Q - MR^{-1}M' = D_1D_1' \geqslant 0$, then $\hat{K}$ exists as the unique positive [non-negative] definite solution of the algebraic Riccati Equation (4.14). Moreover, the resulting closed-loop system (4.4) is asymptotically stable.*

Proof See [W15].

It is noted that the choice of weighting matrices Q, M and R reflects a "trade-off" between penalizing excursions from the zero reference state in $x'Qx$ and the need to limit the required control action by assigning a penalty $u'Ru$. The controllability [stabilizability] condition of Lemma 4.1 is a fundamental one in the light of Theorem 1.15 [Corollary 1.3]. The observability [detectability] condition is necessary to ensure that all of the [unstable] system states contribute to the final value of J. One last comment is that the solution of the optimal linear regulator problem for the linearized perturbation model (1.17) in Chapter 1, by way of minimizing terms quadratic in $\delta x(t)$ and $\delta u(t)$, is consistent with an earlier model validation aim of keeping $\alpha_0(\delta x, \delta u)$ and $\beta_0(\delta x)$ small.

4.3 THE OBSERVER-BASED CONTROLLER

The linear feedback control strategies of Section 4.2 presume the system state vector to be completely measurable. If, on the contrary, the linear system is as in Equations (4.1) and (4.2) with the matrix C of full rank $m < n$, some of the elements of the state vector $x(t) \in R^n$ are inaccessible thereby excluding the possibility of realizing the non-dynamic state feedback controller (4.3). Fortunately, however, as we saw in Section 1.6, linear stabilizing feedback control may still be accomplished by using an estimate $\hat{x}(t)$, generated by a state observer, instead of the unknown state $x(t)$ in (4.3). We now proceed to consider a linear feedback controller using the minimal-order state observer described in Section 1.7. That is, we consider the application of the dynamic

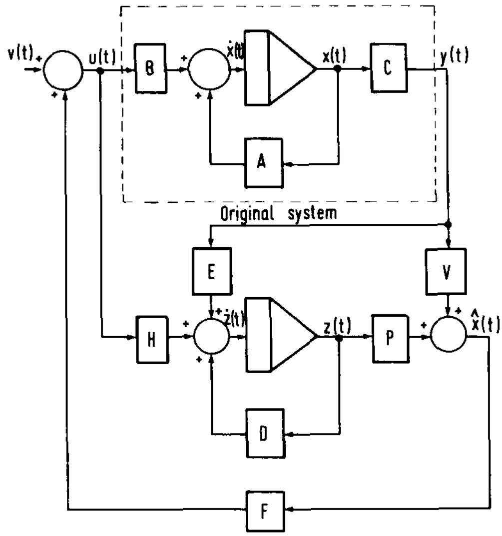

Fig. 4.2 Block diagram of observer-based controller (compensator).

observer-based controller

$$\dot{z}(t) = Dz(t) + Ey(t) + Hu(t), \quad z(t) \in R^{n-m} \tag{4.16}$$

$$\hat{x}(t) = Pz(t) + Vy(t) \tag{4.17}$$

$$u(t) = F\hat{x}(t) + v(t). \tag{4.18}$$

A schematic of the observer-based controller in closed-loop feedback operation is shown in Fig. 4.2.

But first let us analyse the open-loop composite system-observer described by the $(2n - m)$th order system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ EC & D \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ H \end{bmatrix} u(t) \tag{4.19}$$

and

$$\hat{x}(t) = [VC \quad P] \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{4.20}$$

It was shown in Equations (1.81) and (1.75) that the state reconstruction error satisfies

$$e(t) \triangleq \hat{x}(t) - x(t) = P\varepsilon(t) \tag{4.21}$$

where the observer reconstruction error $\varepsilon(t) \triangleq z(t) - Tx(t)$ has the dynamics

$$\dot{\varepsilon}(t) = D\varepsilon(t), \quad \varepsilon(0) = z_0 - Tx_0. \tag{4.22}$$

Use of the invertible Lyapunov transformation

$$\begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -T & I_{n-m} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{4.23}$$

gives rise to the equivalent open-loop system-observer realization

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t). \tag{4.24}$$

In Equation (4.24) is laid bare the important fact that for the composite system-observer, the $n - m$ observer modes are uncontrollable from, and therefore unexcited by, the control input $u(t)$.

Returning to the closed-loop situation, the feedback controller using (4.18) and (4.21) is given by

$$u(t) = Fx(t) + FP\varepsilon(t) + v(t). \tag{4.25}$$

Hence, the overall closed-loop system response may be described, via (4.24) and (4.25), by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A + BF & BFP \\ 0 & D \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t). \tag{4.26}$$

Notice from the upper triangular form of (4.26) that the minimal-order observer does not effect the stability of the original closed-loop system but merely adjoins its own stability properties. Compared to direct state feedback (4.3) it is noted, however, that the closed-loop system under feedback control (4.25) will be degraded in its initial phase by the observer error $\varepsilon(t)$ in (4.22), due to initial state uncertainty.

If we proceed to consider the equivalent class of observers discussed in Section 2.3, the observer coefficient matrices are completely specified by the arbitrary $(n - m) \times m$ gain matrix K, viz.

$$D = A_{22} + KA_{12} \tag{4.27}$$

$$E = A_{21} + KA_{11} - A_{22}K - KA_{12}K \tag{4.28}$$

$$H = B_2 + KB_1 \tag{4.29}$$

$$P = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}, \quad V = \begin{bmatrix} I_m \\ -K \end{bmatrix}. \tag{4.30}$$

Then, the overall closed-loop system response (4.26) becomes

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \hat{A} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t) \tag{4.31}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A + BF & BFP \\ 0 & A_{22} + KA_{12} \end{bmatrix}. \tag{4.32}$$

By virtue of the invertible linear transformation (4.23), the characteristic polynomial of the composite closed-loop system (4.19), (4.18) and (4.20) is the same as that of (4.31), namely,

$$\det\left(\lambda I_{2n-m} - \begin{bmatrix} A + BF & BFP \\ 0 & A_{22} + KA_{12} \end{bmatrix}\right)$$
$$= \det(\lambda I_n - A - BF) \times \det(\lambda I_{n-m} - A_{22} - KA_{12}). \tag{4.33}$$

Corresponding to Theorem 1.17 and Corollary 1.5 for the full-order observer, we have by Lemma 2.1 the following result.

Theorem 4.3 *If the linear system (4.1) and (4.2) is completely controllable and completely observable, there exist gain matrices F and K such that the $2n - m$ eigenvalues of the closed-loop system matrix $\hat{A}$ in (4.32) can be arbitrarily assigned, in particular to positions in the left-half complex plane.*

Corollary 4.1 *If the linear system (4.1) and (4.2) is stabilizable and detectable, there exist gain matrices F and K such that the closed-loop system matrix $\hat{A}$ in (4.32) is a stability matrix, i.e. has all its $2n - m$ eigenvalues on the left-hand side of the complex plane.*

As with the full-order observer of Section 1.6, Theorem 4.3 and Corollary 4.1 display a separation principle in observer-based controller design; namely, the minimal-order state observer (4.16), (4.17), (4.27) to (4.30) and the linear feedback controller (4.18) may be designed separately by independent choice of the gain matrices K and F. In both minimal-order and full-order cases, an observer-controller duality is manifest which allows us to exploit the observer eigenvalue assignment techniques of Section 2.4 and Section 2.5 and the dual control eigenvalue–eigenvector selection method of Theorem 4.1 interchangeably to full advantage. We forbear to do this but shall extend the accessible state feedback result of Theorem 4.2 to embrace the design of optimal dynamical controllers using a minimal-order observer in Section 4.4.

At this point it is worthwhile to pause and ponder further the choice of stabilizing observer gain matrix K. A fast observer error transient response engendered by a very high value of gain is not as attractive as it might seem at first sight; for, in the limit, the observer acts as a differentiation device with the consequence that it is highly susceptible to inevitably present measurement noise. To elucidate further, suppose the actual system (4.1) and (4.2) is subject to random disturbances $\xi(\cdot)$ and $\eta(\cdot)$; that is, it may be described by

$$\dot{x}(t) = Ax(t) + Bu(t) + \xi(t) \tag{4.34}$$

$$y(t) = Cx(t) + \eta(t). \tag{4.35}$$

The observer error, $\varepsilon(t) \triangleq z(t) - Tx(t)$, in deploying the deterministic observer (4.16), (4.17), (4.27) to (4.30) for the stochastic system (4.34) and (4.35) is readily shown to satisfy (cf. Equation (4.22))

$$\dot{\varepsilon}(t) = D\varepsilon(t) + E\eta(t) - T\xi(t). \tag{4.36}$$

Notwithstanding the fact that large values of K can drive the expected or mean value of ε speedily to zero, instantaneous values of $E\eta(\cdot)$ and $T\xi(\cdot)$ and hence $\varepsilon(\cdot)$ may be unacceptably large. Accordingly, following the original suggestion of Luenberger [L11] it has become established practice to choose the observer eigenvalues to be not much faster than those of the original closed-loop system matrix $A + BF$. More will be said on this from a transfer–function point of view in Chapter 8. Criteria and techniques for choosing the observer gain in a linear least-square error sense are presented in Chapter 6.

We now continue with a brief look at how the observer-based controller (4.16) to (4.18) may be viewed as a compensator for the system (4.1) and (4.2). Suppose

$$D_1 = D + HFP, \quad E_1 = E + HFV \tag{4.37}$$

$$F_1 = FP, \quad H_1 = FV. \tag{4.38}$$

Then the observer-based controller Equations (4.16) to (4.18) define the *dynamic compensator*

$$\dot{z}(t) = D_1 z(t) + E_1 y(t) + Hv(t), \quad z(0) = z_0 \tag{4.39}$$

$$u(t) = F_1 z(t) + H_1 y(t) + v(t). \tag{4.40}$$

Equations (4.39) and (4.40) constitute a general time–domain description of a dynamic compensator of order l ($z(t) \in R^l$), $0 \leqslant l \leqslant n - m$. Like the observer-based controller, depicted in the block diagram of Fig. 4.2, it operates dynamically on the available system measurements $y(t)$, $t \in [0, t_1]$ to produce a control input $u(t)$, $t \in [0, t_1]$ that (hopefully) stabilizes the system (4.1) and (4.2). At the price of maximum order $l = n - m$, we have just seen that in the observer-based controller (4.16) to (4.18), this desirable property is assured. At the bottom of the scale, it may be possible to stabilize the system by using the structurally simple zero-order ($l = 0$) or instantaneous output feedback controller.

$$u(t) = H_1 y(t) + v(t). \tag{4.41}$$

Casting our mind back to Theorem 4.3 and Corollary 4.1, the reduction in order of the observer by m, the number of measurable states in the output

$y(t) \in R^m$ of (4.2), meant that only $2n - m$ eigenvalues of the composite closed-loop system needed assignment. A further reduction in the number of eigenvalues requiring assignment is to be had by taking notice of the dual fact that max (r, m) eigenvalues of the closed-loop system may be directly assigned by the non-dynamic output feedback law (4.41) in accordance with Lemma 4.2 [D3].

Lemma 4.2 *Given the completely controllable and observable linear system (4.1) and (4.2), an $(r \times m)$ output feedback gain matrix H_1 can always be found such that $A + BH_1C$ has q eigenvalues arbitrarily close to q specified symmetric values, with $0 \leqslant q \leqslant \max(r, m)$.*

If, by Lemma 4.2, $r - 1$ eigenvalues of the system (4.1) are *a priori* arbitrarily assigned by the output feedback law (4.41), an observer-based controller of reduced dimension $l - m$ is required to control the remaining subsystem of order $l = n - (r - 1)$. Thus, corresponding to Theorem 4.3 we have the following reduction in the eigenvalue assignment problem.

Theorem 4.4 *If the linear system (4.1) and (4.2) is completely controllable and completely observable, there exists a dynamic observer-based controller of order $n + 1 - m - r$ such that the $2n + 1 - m - r$ symmetrical eigenvalues of the composite closed-loop system can be arbitrarily assigned.*

4.4 AN OPTIMAL OBSERVER-BASED CONTROLLER

As an alternative to eigenvalue–eigenvector assignment methods of observer-based controller design, consideration is now focussed on the extension of the optimal linear regulator solution of Theorem 4.2 to systems with inaccessible state. Again we demonstrate a separation of observer and linear feedback controller design, this time by way of formulating and solving the following fixed configuration optimization problem.

Given the linear observer-based controller described by (4.16) to (4.18), (4.27) to (4.30) in which $v(t) \equiv 0$, and the initial state $z_0 \in R^{n-m}$, find F^*, z_0^* and K^* (if they exist) such that

$$J^* \triangleq J(F^*, z_0^*, K^*) \leqslant J(F, z_0, K) \tag{4.42}$$

for all arbitrary F, z_0 and K where the performance index is the expectation of (4.11)

$$J = E\left[\int_0^\infty (x'(t)Qx(t) + 2x'(t)Mu(t) + u'(t)Ru(t))\,dt\right] \tag{4.43}$$

and the expectation $E[\cdot]$ is taken over all random initial states of the system (4.1). It is further assumed that the system (4.1) and (4.2) is completely controllable [stabilizable] and completely observable [detectable], and that the initial state x_0 of (4.1) is an unknown random vector which may, however, be described by the first and second order statistical moments

$$E[x_0] = \bar{x}_0 \tag{4.44}$$

$$E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'] = \Sigma_0 = \Sigma_0' > 0. \tag{4.45}$$

Substituting (4.25) into (4.43) with $v(t) \equiv 0$, one has

$$J = E\left[\int_0^\infty [x'(t), \varepsilon'(t)]\hat{Q}\begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} dt\right] \tag{4.46}$$

where

$$\hat{Q} \triangleq \begin{bmatrix} Q + MF + F'M' + F'RF & F'RFP + MFP \\ P'F'RF + P'F'M' & P'F'RFP \end{bmatrix}. \tag{4.47}$$

Also, the solution of the homogeneous system of Equations (4.31) with $v(t) \equiv 0$ is

$$\begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} = \exp(\hat{A}t)\begin{bmatrix} x_0 \\ \varepsilon_0 \end{bmatrix}. \tag{4.48}$$

Since the resulting system must, by assumption, be asymptotically stable to retain a meaningful optimization problem, the cost (4.46) may be rewritten, using (4.48), as

$$J = E\left[[x_0', \varepsilon_0']L\begin{bmatrix} x_0 \\ \varepsilon_0 \end{bmatrix}\right] \tag{4.49}$$

where

$$L = \int_0^\infty \exp(\hat{A}'t)\hat{Q} \exp t(\hat{A}t)\, dt \tag{4.50}$$

is the symmetric positive semi-definite solution [B7] of the algebraic equation

$$\hat{A}'L + L\hat{A} + \hat{Q} = 0. \tag{4.51}$$

By Equations (4.44), (4.45), (4.22) and (4.49) and the fact that (Appendix A) trace $XY =$ trace YX,

$$\begin{aligned} J &= \operatorname{trace} L\hat{\Sigma} \\ &= \operatorname{trace}\begin{bmatrix} L_{11} & L_{12} \\ L_{12}' & L_{22} \end{bmatrix} \\ &\qquad \times \begin{bmatrix} \Sigma_0 + \bar{x}_0\bar{x}_0' & -\Sigma_0 T' + \bar{x}_0(z_0 - T\bar{x}_0)' \\ -T\Sigma_0 + (z_0 - T\bar{x}_0)\bar{x}_0' & T\Sigma_0 T' + (z_0 - T\bar{x}_0)(z_0 - T\bar{x}_0)' \end{bmatrix} \end{aligned} \tag{4.52}$$

where

$$\hat{\Sigma} \triangleq E\left[\begin{bmatrix} x_0 \\ \varepsilon_0 \end{bmatrix}[x_0', \varepsilon_0']\right]. \tag{4.53}$$

Note the explicit dependence of the cost upon the choice of observer initial condition z_0.

The specific optimal control problem of (4.42) can alternatively be stated as one of finding F, z_0, K and L so as to minimize $J(F, z_0, K, L)$ subject to the algebraic constraint (4.51).

This constrained optimization problem can now be reformulated as the problem, with no constraints, of finding F, z_0, L and Λ so as to minimize the Lagrangian

$$\Phi = \operatorname{tr} L\hat{\Sigma} + \operatorname{tr}[(\hat{A}'L + L\hat{A} + \hat{Q})\Lambda'] \tag{4.54}$$

where Λ is a $(2n - m) \times (2n - m)$ matrix of Lagrange multipliers given by

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \tag{4.55}$$

and Σ_0 is suitably partitioned as

$$\Sigma_0 = [\Sigma_1, \Sigma_2] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix}. \tag{4.56}$$

Necessary conditions for optimality are

$$\frac{\partial\Phi}{\partial F} = 0, \quad \frac{\partial\Phi}{\partial z_0} = 0, \quad \frac{\partial\Phi}{\partial K} = 0, \quad \frac{\partial\Phi}{\partial L} = 0, \quad \frac{\partial\Phi}{\partial\Lambda} = 0. \tag{4.57}$$

More specifically, using the gradient notation contained in Appendix A,

$$\frac{\partial\Phi}{\partial F} = (B'L_{11} + M' + RF)(\Lambda_{11} + \Lambda_{12}P') + (B'L_{12} + RFP)$$
$$\times (\Lambda_{21} + \Lambda_{22}P') = 0 \tag{4.58}$$

$$\frac{\partial\Phi}{\partial z_0} = L_{12}'\bar{x}_0 + L_{22}(z_0 - T\bar{x}_0) = 0 \tag{4.59}$$

$$\frac{\partial\Phi}{\partial K} = L_{12}'[\Lambda_{12}A_{12}' - [\Sigma_0 + \bar{x}_0\bar{x}_0']_1]^* + L_{22}[\Lambda_{22}A_{12}' - [\bar{z}_0\bar{x}_0']_1$$
$$+ [\Sigma_0 + \bar{x}_0\bar{x}_0']_{21} + K[\Sigma_0 + \bar{x}_0\bar{x}_0']_{11}] = 0 \tag{4.60}$$

---

* $[X]_1$ and $[X]_{11}$, etc. refer to the appropriate partitions of the matrix

$$X = [X_1, X_2] = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$\frac{\partial \Phi}{\partial L} = \hat{\Sigma} + \Lambda \hat{A}' + \hat{A}\Lambda = 0 \tag{4.61}$$

$$\frac{\partial \Phi}{\partial \Lambda} = \hat{A}'L + L\hat{A} + \hat{Q} = 0. \tag{4.62}$$

It is observed that Equation (4.62) is, as expected, identical to (4.51).

4.4.1 A candidate optimal controller

We now seek an observer-based controller which will satisfy the necessary conditions for optimality (4.58) to (4.62).

Recall from Section 4.2 that when the state of the system (4.1) is completely available the optimal controller is the linear instantaneous feedback control law (4.12) to (4.14). If the state of the system (4.1) is not completely available it can, however, by Equations (4.22) and (4.21), be reconstructed exactly using an observer if the initial state is known and the observer initial condition is set as

$$z_0 = Tx_0. \tag{4.63}$$

The optimal controller is, as before, given by (4.12) to (4.14).

Finally, if as is the problem here, the initial state is an unknown random vector with known statistics (4.44) and (4.45), the optimal observer initial condition might, plausibly, be the expected value

$$z_0 = T\bar{x}_0. \tag{4.64}$$

These considerations, together with the nature of Equations (4.58) to (4.62) suggest that the following observer-based controller will satisfy the necessary conditions for optimality:

$$F^* = -R^{-1}(B'L^*_{11} + M') \tag{4.65}$$

$$z^*_0 = [K, I_{n-m}]\bar{x}_0 \tag{4.66}$$

$$K^* = -(\Sigma_{21} + \Lambda^*_{22}A'_{12})\Sigma^{-1}_{11} \tag{4.67}$$

$$L^*_{12} = 0 \tag{4.68}$$

$$\Lambda^*_{12} = -P\Lambda^*_{22}. \tag{4.69}$$

Indeed, routine manipulation reveals that (4.58) to (4.60) are satisfied while (4.61) and (4.62) reduce to

$$\begin{aligned} 0 = A_{22}\Lambda^*_{22} + \Lambda^*_{22}A'_{22} + \Sigma_{22} \\ - (\Sigma_{21} + \Lambda^*_{22}A'_{12})\Sigma^{-1}_{11}(\Sigma_{21} + \Lambda^*_{22}A'_{12})' \end{aligned} \tag{4.70}$$

$$0 = A'L^*_{11} + L^*_{11}A + Q - (L^*_{11}B + M)R^{-1}(L^*_{11}B + M)' \tag{4.71}$$

$$0 = (A + BF^*)\Lambda_{11}^* + \Lambda_{11}^*(A + BF^*)' + \Sigma_0 + \bar{x}_0\bar{x}_0' - (BF^*P)\Lambda_{22}^*P' - P\Lambda_{22}^*(BF^*P)' \tag{4.72}$$

$$0 = (A_{22} + K^*A_{12})'L_{22}^* + L^*_{22}(A_{22} + K^*A_{12}) + P'F^{*\prime}RF^*P \tag{4.73}$$

It should be noted that satisfaction of the necessary conditions for optimality by (4.65) to (4.69) is subject to both the existence of a solution to each of the algebraic matrix Riccati Equations (4.70) and (4.71) and the asymptotic stability of the overall system (4.31).

Sufficient conditions for the existence of a unique L_{11}^* (and hence, by (4.65), of F^*), as well as for the asymptotic stability of $(A + BF^*)$, are presented in Lemma 4.1. Consequently, if by Lemma 4.1 $(A + BF^*)$ is a stability matrix, the overall system (4.31) will remain asymptotically stable if, and only if, there exists an asymptotically stable observer matrix $(A_{22} + K^*A_{12})$. By the dual of Lemma 4.1 we have:

Lemma 4.3 *If the pair (A_{22}, A_{12}) is completely observable [detectable] and the pair $(A_{22} - \Sigma_{21}\Sigma_{11}^{-1}A_{12}, D_2)$ is completely controllable, [stabilizable], where $\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}' \equiv D_2D_2' \geqslant 0$, then Λ_{22}^* exists as the unique positive [non-negative] definite solution of the algebraic matrix Riccati Equation (4.70). Moreover, the resulting observer matrix $(A_{22} + K^*A_{12})$ is asymptotically stable.*

Corollary 4.2 *If $\Sigma_0 > 0$, a sufficient condition for the results of Lemma 4.3 to hold, and in particular for the asymptotic stability of the observer matrix $A_{22} + K^*A_{12}$, is that (A_{22}, A_{12}) be an observable (detectable) pair.*

Proof Using the formula for the inverse of a partitioned matrix (Appendix A), $\Sigma_0 > 0$ implies that

$$\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}' = D_2D_2' > 0.$$

Therefore

$$W(\tau, t) = \int_\tau^t \exp[(A_{22} - \Sigma_{21}\Sigma_{11}^{-1}A_{12})\lambda]D_2D_2' \times \exp[(A_{22} - \Sigma_{21}\Sigma_{11}^{-1}A_{12})'\lambda]\,d\lambda$$

is positive-definite or by Theorem 1.2 the pair $(A_{22} - \Sigma_{21}\Sigma_{11}^{-1}A_{12}, D_2)$ is completely controllable. Since the conditions of Lemma 4.3 are therefore satisfied, the corollary immediately follows. Q.E.D. □

Corollary 4.2 corresponds to the conditions assumed in this section. Also,

the observability (detectability) of (A_{22}, A_{12}) is ensured by that of (A, C) in Lemma 2.1.

In summary, we have obtained a (stable) observer-based controller which satisfied the necessary conditions for optimality (4.58) to (4.62). Reviewing Equations (4.65) to (4.73), a *separation principle* is seen to hold. In particular, the control law (4.18) is completely determined by (4.65) with L_{11}^* computed from (4.71) and, as such, is identical to that given by (4.13) and (4.14) when the state is completely available. Furthermore, the observer is separately completely specified by Equations (4.66) and (4.67) where Λ_{22}^* is computed from (4.70). Therefore, the observer may be realized by Equations (4.16), (4.17), (4.27) to (4.30) and initial condition (4.66). So, as a result of the separation or "decoupling" of control and state reconstruction, the realizations of control and observer are completely independent.

4.5 A TRANSFER–FUNCTION APPROACH TO OBSERVER-BASED CONTROLLER DESIGN

So far our discussions of observers and their dynamic role in linear feedback control systems design has been based exclusively on *internal* state–space models in the time–domain. Attention is now focussed on developing *external* input–output properties of such systems using the transfer–function description of linear multivariable state–space systems of Section 1.4. There it is recalled that the transfer–function matrix $T(s)$ relating the input transform $u(s)$ to the output transform $y(s)$ of the open-loop system (4.1) and (4.2) is given by

$$T(s) = C(sI - A)^{-1}B \tag{4.74}$$

$$= C\frac{\operatorname{adj}(sI - A)B}{\det(sI - A)}. \tag{4.75}$$

4.5.1 Linear state feedback

Before embarking on a transfer–function matrix analysis of multivariable observer-based controllers, it is of interest to first consider that of linear state feedback. Suppose the linear feedback control law (4.3) or

$$u(t) = Fx(t) + v(t) \tag{4.76}$$

is applied to the open-loop system (4.1) and (4.2) of which the open-loop transfer–function matrix is $T(s)$. In respect of the closed-loop system configuration depicted in Fig. 4.1, it is readily seen from Equation (4.4) that the closed-loop transfer–function matrix under linear state feedback, $T_{SF}(s)$, is

given by

$$T_{SF}(s) = C(sI - A - BF)^{-1}B \tag{4.77}$$

or

$$T_{SF}(s) = \frac{C \operatorname{adj}(sI - A - BF)B}{\det(sI - A - BF)}. \tag{4.78}$$

Consider now the characteristic polynomial of the closed-loop system:

$$\begin{aligned}\det(sI - A - BF) &= \det\{(sI - A)[I_n - (sI_n - A)^{-1}BF]\}\\ &= \det(sI - A) \times \det[I_n - (sI_n - A)^{-1}BF]\\ &= \det(sI - A) \times \det[I_r - F(sI_n - A)^{-1}B]\end{aligned} \tag{4.79}$$

upon using the identity $\det[I_n - AB] = \det[I_m - BA]$ of Appendix A. In Equation (4.79)

$$F_1(s) \triangleq I_r - F(sI_n - A)^{-1}B \tag{4.80}$$

is known as the system *return-difference matrix* referred to the point N in the feedback configuration of Fig. 4.1. The name is almost self-explanatory. Suppose all feedback loops are broken at point N in Fig. 4.1 and that a signal transform vector $\alpha(s)$ is injected at this point. The transform of the returned signal at N is then

$$F(sI_n - A)^{-1}B\alpha(s)$$

and the difference between the injected and returned signals is thus

$$[I_r - F(sI_n - A)^{-1}B]\alpha(s) = F_1(s)\alpha(s) \tag{4.81}$$

where $F_1(s)$ is the system return-difference matrix defined in (4.80). Also, from (4.79) and (4.80) we have that

$$\det F_1(s) = \frac{\det(sI_n - A - BF)}{\det(sI_n - A)} \tag{4.82}$$

$$= \frac{\text{closed-loop characteristic polynomial}}{\text{open-loop characteristic polynomial}}. \tag{4.83}$$

The return-difference system description will be considered further in the dual context of observer design in Chapter 8.

4.5.2 Linear state estimate feedback

Let us now return to the basic problem under discussion; that of observer-based controller design wherein a linear feedback control law in the observer state estimate (4.18) is used instead of (4.76). It is recalled from Equation (4.31)

of Section 4.3 that the overall closed-loop system response is described by the $(2n - m)$th order system of equations:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A + BF & BFP \\ 0 & A_{22} + KA_{12} \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t) \tag{4.84}$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix}. \tag{4.85}$$

Treating the composite system (4.84) and (4.85) in much the same way as we did the original system (4.1) and (4.2) under feedback (4.76), it immediately follows that the overall closed-loop transfer–function matrix $T_{0C}(s)$ of the observer-based control system, between $y(s)$ and $v(s)$, is given by

$$T_{0C}(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI_n - A - BF & -BFP \\ 0 & sI_{n-m} - A_{22} - KA_{12} \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{4.86}$$

or, upon evaluation of the matrix inverse (Appendix A),

$$T_{0C}(s) = C(sI_n - A - BF)^{-1} B \frac{\det (sI_{n-m} - A_{22} - KA_{12})}{\det (sI_{n-m} - A_{22} - KA_{12})}. \tag{4.87}$$

Equation (4.87) bears testimony to the remarkable fact that the transfer–function matrix of a closed-loop system incorporating a state observer (of minimal order or otherwise) is *identical* to the transfer function matrix of a closed-loop system employing direct state feedback: viz., $T_{0C}(s) \equiv T_{SF}(s)$. The cancellable polynomial $\det (sI_{n-m} - A_{22} - KA_{12})$ corresponds to the fact, originally noted by Luenberger [L11], that the error dynamics of the observer are uncontrollable from u or v (see also Equation (4.24)). Our main concern is that this cancellable polynomial $\det (sI_{n-m} - A_{22} - KA_{12})$ be a stable one and this we can ensure by appropriate choice, independent of the control gain F, of the observer gain matrix K.

4.6 NOTES AND REFERENCES

Original and fundamental studies of linear multivariable feedback control from a state–space point of view, of which Section 4.2 carries on from Section 1.5, were undertaken by Kalman in a series of highly influential articles [K6], [K8], [K10], [K11]. These papers are widely regarded [M5] as marking the beginning of control studies as a mature scientific discipline. The eigenvalue–eigenvector (eigenstructure) assignment result of Theorem 4.1, implicit in Kimura [K21], is due to Moore [M15], [K26]; see also Porter and D'Azzo [P15] and Fahmy and O'Reilly [F2], [F3].

Theorem 4.2 and Lemma 4.1 constitute the first general and rigorous

solution of a long outstanding problem of control, the linear quadratic control problem, from a state–space point of view by Kalman [K8]. The simple least-squares proof of Theorem 4.2 is along the lines of Brockett [B18]. Of the voluminous literature on the optimal regulator, dual Kalman-Bucy filter, allied matrix Riccati equations and related topics that have appeared in its wake, the general reader is referred to [A8], [A14] and [K4]. Special mention is made, however, of the fact that not only does the linear quadratic state feedback regulator stabilize unstable systems, realize a prescribed transient response but it is also robust against variations in open-loop dynamics [S1]. The selection of the cost-weighting matrices to comply with pre-specified design requirements is a non-trivial exercise. A review of this problem, of the first importance in applications, together with a discussion of related asymptotic properties as the control weighting matrix approaches the null matrix is to be found in Stein [S19].

The substance of Theorem 4.3 and Corollary 4.1 to the effect that an observer does not alter the eigenvalues of the original closed-loop system, but merely adjoins its own eigenvalues is contained in the original work of Luenberger [L8], [L9]. Related lower order dynamic compensator and static output feedback design are introduced respectively by Brasch and Pearson [B17] and Davison [D2]. For more recent progress on these problems, including that of compensator minimal order *vs* eigenvalue assignability, see Kimura [K21], [K22] and the survey by Munro [M22]. Theorem 4.4 is patterned on the recent extension by Balestrino and Celentano [B1].

The problem of linear quadratic optimization of the overall observer based controller in Section 4.4 was first considered by Newmann [N5], [N8]; see also Porter and Woodhead [P16]. Our derivation is fashioned after the development in Millar [M11] and Rom and Sarachik [R6], [R7]. The latter authors solved a slightly less general problem in that they took the observer initial condition to be zero rather than retain it as an arbitrary design parameter in the overall optimization problem. On the other hand, Meada and Hino [M9] did retain the observer initial condition but dropped that of the controller feedback gain matrix by assuming *a priori* (and correctly as it turns out) that the optimal feedback gain matrix was the same as that which arises for the regulator with complete state information available. Discrete-time solutions are presented independently by O'Reilly and Newmann [O26] and Kumar and Yam [K40] while further refinements are reported in [K38], [G5], [W6]. A specific optimal controller, based on a dual minimal-order observer discussed in Section 1.9, is developed by Blanvillain and Johnson [B11].

Instead of using the design freedom which remains after eigenvalue assignment to specify eigenvectors, it may be used to minimize a quadratic performance index as previously discussed [R1], [A1] or to minimize observer

steady-state errors induced by system parameter variations [R2], [T3]. Further light [M12], [M13] is thrown on the observation, originally made by Bongiorno and Youla [B12], [B13], that for some initial errors the cost increase of an optimal regulator incorporating an observer tends either to infinity or zero. Related optimal fixed-configuration compensator design for continuous-time and discrete-time systems is reviewed by Mendal and Feather [M10] and O'Reilly [O14] respectively. A caveat on the procedure of averaging the performance index over initial states for the output feedback case [A5] has implications for the optimal compensator and observer design problem as well.

Inability to alter system zeros in undesirable locations, for example from phase–margin considerations, may be regarded as a limitation of state feedback. Also, referring back to the canonical structure theorem (Theorem 1.11, it can be shown [W10] that completely uncontrollable or completely unobservable modes of the state–space system correspond to pole–zero cancellations in the transfer function. This corroborates the well-known frequency–domain view, extended to multi-input systems, that unstable pole–zero cancellations are undesirable as regards closed-loop system stability. It is of historical interest to note that it was the discussion of hidden oscillations and instability in sampled-data systems by Jury [J8] which stimulated the formulation of the canonical decomposition theorem by Kalman [K10]. A study of the fundamental role poles and zeros have to play in linear multivariable state–space systems and linear polynomial-matrix systems is conducted in Chapter 8 and Chapter 9 respectively.

Although treated for the most part in the literature from a time–domain standpoint, additional insight into the mechanism of observer-based system compensation is gained by pursuing a complementary transfer–function approach. Multi-input generalizations of the classic Bode feedback concept of return difference, considered in Section 4.5.1, are discussed by MacFarlane [M3] and Perkins and Cruz [P5]. Looked at from a frequency–domain point of view, the advantage of state feedback is that it provides phase advance, in the shape of system zeros, to counteract the lagging effects of high-loop gains [M2]. Transfer–function matrix analysis and design of an open-loop observer seems to have been first considered by Retallack [R3]. The closed-loop transfer–function matrix analysis of a feedback control system incorporating a minimal-order state observer, introduced in Section 4.5.2, is largely influenced by Wolovich [W10]. A transfer–function analysis of the discrete accessible and inaccessible state feedback control problem, through the medium of the z-transform, is algebraically equivalent to the continuous-time one presented here [R10]. Further exploration of observer-based controller compensation in the frequency domain for both state–space and transfer–function models is made in Chapter 8 and Chapter 9.

Chapter 5

Minimum-Time State Reconstruction of Discrete Systems

5.1 INTRODUCTION

This chapter considers the problem of observer design for the minimum-time state and linear state function reconstruction of discrete-time linear systems. The minimum-time reconstruction problem is one of reconstructing the system state vector (or linear state function), on the basis of past and present system inputs and outputs, in a minimum number of time steps. As intimated in Chapter 1, the possibility of such state reconstruction is closely linked to the related notions of system observability and system reconstructibility. These two criteria underpin the problem formulation of Section 5.2 and the two minimum-time full-order state observer designs of Section 5.3.

The first design method, based on system reconstructibility, derives from formulating the problem as a dual deadbeat control problem. The second method exploits the fact that deadbeat observer design is equivalent to the assignment of the eigenvalues of the observer coefficient matrix to the origin in the complex plane. In this way, a simple effective minimum-time full-order observer is derived by first transforming the system into an observable companion form. An important feature of both methods is an adequate selection procedure for choosing n linearly independent columns of the system reconstructibility (observability) matrix to form a basis for the error vector space. In each method the full-order observer is specified by a gain matrix in compact form and the system state vector can be reconstructed in a minimum of steps equal to ν, the observability index of the system.

On the basis of these full-order observer designs, two minimal-order observers of simpler structure are developed in Section 5.4. These minimal-order observers are specified by a gain matrix in compact form, and it is

observed that the system state may be reconstructed in $\nu - 1$ time steps. A third method of minimal-order state observer design involves zero-value eigenvalue assignment for systems in observable companion form and, as a consequence, may be regarded as a special case of that encountered in Section 2.5.

Section 5.5 discusses the inaccessible state deadbeat control problem or the problem of driving a system from any arbitrary initial state to the origin in a minimum number of time steps, assuming the system state is only partially available. The inaccessible state deadbeat control problem can be solved by first reconstructing the system state and then designing a deadbeat controller on the basis of linear feedback of the reconstructed state. This latter task is accomplished by recognizing that the deadbeat control problem is the mathematical dual of the minimum-time full-order observer problem. Alternatively, rather than having to first reconstruct the complete state vector, minimum-time reconstruction of a linear state function such as a control law can be fulfilled by a linear function observer, often of dimension substantially lower than that of a minimal-order state observer. Emphasis in Section 5.6 is on minimizing the requisite number of time steps through working with the generalized notion of linear state function reconstructibility.

5.2 THE MINIMUM-TIME STATE RECONSTRUCTION PROBLEM

Consider the discrete-time linear time-invariant system

$$x(k + 1) = Ax(k) + Bu(k) \tag{5.1}$$

$$y(k) = Cx(k) \tag{5.2}$$

where $x(k) \in R^n$ is the state vector, $u(t) \in R^r$ is the control vector, $y(k) \in R^m$ is the system output vector and the output matrix C is of full rank $m < n$. As in Definition 1.5, the system is completely state reconstructible if the state can be determined in a finite number of steps from past and present inputs and outputs. In fact, by Theorem 1.9, the system (5.1) and (5.2) is completely state reconstructible in at most ν steps where ν is the system observability index associated with the condition (1.38).

A full-order observer which will reconstruct the state of the system (5.1) and (5.2) is described by the nth order system

$$\hat{x}(k + 1) = A\hat{x}(k) + Bu(k) + G(C\hat{x}(k) - y(k)) \tag{5.3}$$

where $\hat{x}(k) \in R^n$ is an estimate of the system state $x(k)$. By (5.1) and (5.3), the state reconstruction error

$$e(k) \triangleq \hat{x}(k) - x(k) \tag{5.4}$$

propagates according to

$$e(k+1) = (A + GC)e(k), \quad k = 0, 1, \ldots \tag{5.5}$$

so that at any instant $q \geqslant 0$

$$e(q) = (A + GC)^q e(0). \tag{5.6}$$

Now it has been remarked that the system state may be reconstructed in not more than v steps. Thus the minimum-time state reconstruction problem is one of finding an observer gain matrix G such that

$$e(v) = 0 \tag{5.7}$$

for any initial state reconstruction error $e(0)$. In other words, it is required to find a matrix G so that $(A + GC)$ is a nilpotent matrix [G1] of index of nilpotency v. Two methods of construction of a deadbeat observer gain G are the subject of Section 5.3.

In the first method, the problem of finding an $n \times m$ gain matrix G such that the closed-loop observer matrix $A + GC$ is a nilpotent matrix is regarded as equivalent to the requirement that the eigenvalues of $A + GC$ are all zero [G1]. Thus the problem may be construed as a special case of the arbitrary eigenvalue assignment problem of Section 1.6 in which, by Theorem 1.16 observability in the sense of (1.38) is equivalent to arbitrary eigenvalue assignment. Furthermore, zero value eigenvalue assignment, or in other words, minimum-time state reconstruction, may be more readily achieved by transformation into the observable companion form [L12] of Section 2.5. This forms the basis of our first minimum-time observer derivation in Section 5.3.

A second method for constructing the minimum-time observer gain matrix G is based on system state reconstructibility instead of system observability. Recall from Theorem 1.9 that condition (1.38) is both necessary and sufficient for state reconstructibility provided $A(\cdot)$ is non-singular, in which case it is equivalent to the condition

$$\operatorname{rank}\left[A'^{-1}C', A'^{-2}C', \ldots, A'^{-v}C'\right] = n. \tag{5.8}$$

Also, in view of the duality between state feedback controllers and state observers we have already witnessed in Chapter 1, it is convenient to recast the minimum-time state reconstruction problem as a dual deadbeat control problem; namely, find the matrix G' such that $A' + C'G'$ is a nilpotent matrix of index of nilpotency v. Then, substituting A for A', B for C', F for G' and

rewriting (5.8) as the ν-step controllability criterion

$$\operatorname{rank}\,[A^{-1}B, A^{-2}B, \ldots, A^{-\nu}B] = n \tag{5.9}$$

we have the following *dual deadbeat control problem*: find a (linear state feedback) control sequence $\mathscr{U} = \{u(k), u(k+1), \ldots, u(k-\nu-1)\}$ such that the state of (5.1) at any instant k is transferred from $x(k)$ to the origin in at most ν time-steps. A selection procedure, useful for choosing n linearly independent columns of (5.9), is provided in the following definition [O20].

Definition 5.1 Let $\{A^{-q}B\}$ be a sequence of $n \times r$ matrices. Let

$$r_q = \operatorname{rank}\,[A^{-1}B, \ldots, A^{-q}B] - \operatorname{rank}\,[A^{-1}B, \ldots, A^{-(q-1)}B].$$

An ordered selection for $\{A^{-q}B\}$ is a sequence of matrices $\{D_q\}$ with $D_q r \times r_q$ for which

$$\operatorname{range}\,[A^{-1}BD_1, \ldots, A^{-q}BD_q] = \operatorname{range}\,[A^{-1}B, \ldots, A^{-q}B] \tag{5.10}$$

for each q. The matrix $[A^{-1}BD_1, \ldots, A^{-q}BD_q]$ has full rank and therefore every $x \in \operatorname{range}\,[A^{-1}B, \ldots, A^{-q}B]$ has a unique decomposition

$$x = A^{-1}BD_1v_1 + \cdots + A^{-q}BD_qv_q. \tag{5.11}$$

The solution to the deadbeat control problem and, by duality, the minimum-time state reconstruction problem is presented in Section 5.3.

5.3 FULL-ORDER MINIMUM-TIME STATE OBSERVERS

The first minimum-time full-order observer derivation is based on zero-value eigenvalue assignment for systems in observable companion form (cf. Section 2.5). Specifically, by means of the state transformation $x = M\bar{x}$, the system (5.1) and (5.2) may be transformed into the observable companion form

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) \tag{5.12}$$

$$y(k) = \bar{C}\bar{x}(k) \tag{5.13}$$

where $\bar{A} \triangleq M^{-1}AM$ and $\bar{C} \triangleq CM$ are given by

$$
\begin{array}{c}
\bar{A} = \left[\begin{array}{ccccc|cc|ccccc}
0 & 0 & \cdots & 0 & x & & x & & & & & x \\
1 & 0 & \cdots & 0 & x & \cdots & x & & & \bigcirc & & x \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & & & & \vdots \\
0 & 0 & \cdots & 1 & x & & x & & & & & x \\
\hline
& & \cdot & & x & & x & & & \cdot & & \\
& & \cdot & & x & \ddots & x & & & \cdot & & \\
& & \cdot & & & & \cdot & & & \cdot & & \\
\hline
& & & & x & & x & 0 & 0 & \cdots & 0 & x \\
& \bigcirc & & & x & \cdots & x & 1 & 0 & \cdots & 0 & x \\
& & & & \vdots & & & \vdots & \vdots & & \vdots & \vdots \\
& & & & x & & x & 0 & 0 & \cdots & 1 & x
\end{array}\right] \\
\qquad \longleftarrow d_1 \longrightarrow \uparrow \sigma_1 \qquad\qquad \longleftarrow d_m \longrightarrow \uparrow \sigma_m
\end{array}
\tag{5.14}
$$

$$
\begin{array}{c}
\bar{C} = \left[\begin{array}{ccc|cccc|c|cccc}
0 & \cdots & 1 & 0 & \cdots & 0 & 0 & & 0 & \cdots & 0 & 0 \\
0 & \cdots & x & 0 & \cdots & 0 & 1 & & 0 & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & x & 0 & \cdots & 0 & x & & 0 & \cdots & 0 & 1
\end{array}\right]. \\
\qquad \longleftarrow d_1 \longrightarrow \uparrow \sigma_1 \quad \longleftarrow d_2 \longrightarrow \uparrow \sigma_2 \qquad \longleftarrow d_m \longrightarrow \uparrow \sigma_m
\end{array}
\tag{5.15}
$$

In particular, $\sum_{i=1}^{m} d_i = n$, max $\{d_i\}$ = the observability index ν. $\sigma_i = \sum_{j=1}^{i} d_j$, $i = 1, 2, \ldots, m$, denote the positions of the non-trivial columns of $\bar{A}$ and $\bar{C}$. The construction of the transformation matrix M is as in Section 2.5. Let the non-trivial columns of $\bar{A}$ and $\bar{C}$ be given by the $n \times m$ matrix $\bar{A}_m$ and the lower triangular $m \times m$ matrix $\bar{C}_m$. Then, in the transformed co-ordinate system, a simple and elegant method of zero value eigenvalue assignment is to choose the observer gain matrix $\bar{G}$ so that $\bar{A} + \bar{G}\bar{C}$ has zero columns in all non-trivial columns. Thus it is required that all non-trivial columns of $\bar{A} + \bar{G}\bar{C}$ equate to zero; that is

$$\bar{A}_m + \bar{G}\bar{C}_m \equiv 0 \tag{5.16}$$

whence

$$\bar{G} = -\bar{A}_m \bar{C}_m^{-1}. \tag{5.17}$$

Also, since

$$\bar{A} + \bar{G}\bar{C} = \left[\begin{array}{ccccc|cc|ccccc}
0 & 0 & \cdots & 0 & 0 & & 0 & & & & & \\
1 & 0 & \cdots & 0 & 0 & & 0 & & & 0 & & \\
\vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & & & & & \\
0 & 0 & \cdots & 1 & 0 & & 0 & & & & & \\
\hline
 & & \cdot & & 0 & \cdot & 0 & & & \cdot & & \\
 & & \cdot & & 0 & & 0 & & & \cdot & & \\
 & & \cdot & & \vdots & \cdot & \vdots & & & \cdot & & \\
 & & \cdot & & 0 & & 0 & & & \cdot & & \\
\hline
 & & & & & & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & & 0 & & & & 0 & 1 & 0 & \cdots & 0 & 0 \\
 & & & & & \cdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & & & & & & 0 & 0 & 0 & \cdots & 1 & 0
\end{array}\right] \tag{5.18}$$

the minimum polynomial of $\bar{A} + \bar{G}\bar{C}$ is $s^{\max\{d_i\}} = s^{\nu}$ which means that

$$(\bar{A} + \bar{G}\bar{C})^{\nu} = 0. \tag{5.19}$$

Therefore, transforming back to the original system co-ordinates, we have the following minimum-time full-order observer solution.

Theorem 5.1 *If the discrete linear system (5.1) and (5.2) is ν-step observable in the sense of (1.38), the state vector $x(k)$ is reconstructed exactly in ν steps by a full-order observer (5.3) parameterized by the $n \times m$ gain matrix G given by*

$$G = -M\bar{A}_m \bar{C}_m^{-1}. \tag{5.20}$$

Our second construction of a minimum-time observer gain is based on the solution of the dual deadbeat control problem discussed in Section 5.2. Assuming the system (5.1) to be completely ν-step controllable in the sense of (5.9), one obtains, by iterating (5.1), the set of initial states, denoted by $\Gamma_\nu(A, B)$, that can be driven to the origin in ν time-steps.

$$\Gamma_\nu(A, B) = \{x(0) = -[A^{-1}Bu(0) + \cdots + A^{-\nu}Bu(\nu - 1)]\} \tag{5.21}$$

for some $u(0), u(1), \ldots, u(\nu - 1)$. Thus, if one chooses an ordered selection $\{D_\nu\}$

for $\{A^{-\nu}B\}$ according to (5.10), then we may write $x(0)$ as

$$x(0) = -[A^{-1}BD_1, \ldots, A^{-\nu}BD_\nu] \begin{bmatrix} v(0) \\ \\ v(\nu-1) \end{bmatrix} \tag{5.22}$$

where

$$u(0) = D_1 v(0), \quad u(1) = D_2 v(1), \quad \ldots, \quad u(\nu-1) = D_\nu v(\nu-1). \tag{5.23}$$

Rewriting (5.22) in the more compact form

$$x(0) = -S \begin{bmatrix} v(0) \\ \vdots \\ v(\nu-1) \end{bmatrix} \tag{5.24}$$

yields, upon inversion,

$$\begin{bmatrix} v(0) \\ \vdots \\ v(\nu-1) \end{bmatrix} = -S^{-1}x(0) \triangleq \begin{bmatrix} H_1 \\ \vdots \\ H_\nu \end{bmatrix} x(0) \tag{5.25}$$

where

$$S \triangleq [A^{-1}BD_1, \ldots, A^{-\nu}BD_\nu]. \tag{5.26}$$

Observe that

$$-\begin{bmatrix} H_1 \\ \vdots \\ H_\nu \end{bmatrix} S = I_n \tag{5.27}$$

implies that the matrix H_1 is given by

$$H_1[A^{-1}BD_1, \ldots, A^{-\nu}BD_\nu] = [-I_{r_1}, 0_{r_2}, \ldots, 0_{r_\nu}] \tag{5.28}$$

so that, by (5.23), (5.25) and (5.28), the control input $u(0)$ is given by

$$u(0) = D_1 v(0) = D_1 H_1 x(0) \triangleq Fx(0) \tag{5.29}$$

where the $r \times n$ gain matrix F satisfies

$$F[A^{-1}BD_1, \ldots, A^{-\nu}BD_\nu] = [-D_1, 0_{r_2}, \ldots, 0_{r_\nu}]. \tag{5.30}$$

Now, if the system can be taken from the initial state $x(0)$ to the origin in ν steps, it can be taken from the state $x(1)$ to the origin in $\nu - 1$ steps. Thus, similar to (5.22)–(5.24), noting $v(\nu) = 0$, we have

$$x(1) = -S \begin{bmatrix} v(1) \\ \vdots \\ v(\nu) \end{bmatrix} \tag{5.31}$$

$$\begin{bmatrix} v(1) \\ \vdots \\ v(v) \end{bmatrix} = -S^{-1}x(1) = \begin{bmatrix} H_1 \\ \vdots \\ H_2 \end{bmatrix} x(1) \tag{5.32}$$

and

$$u(1) = Fx(1) \tag{5.33}$$

where F satisfies (5.30). In like manner, it is readily shown that, in general,

$$u(k) = Fx(k), \quad k = 0, 1, \ldots, v - 1 \tag{5.34}$$

where F satisfies (5.30), is the desired v-step deadbeat controller. Then, by duality as discussed in Section 5.2, a second minimum-time state reconstruction solution is given by the following Theorem.

Theorem 5.2 *If the discrete linear system (5.1) and (5.2) is v-step state reconstructible in the sense of (5.8), the state vector $x(k)$ is reconstructed exactly in v steps by a full-order observer (5.3) parameterized by the $n \times m$ gain matrix G satisfying*

$$G'[A'^{-1}C'D_1, \ldots, A'^{-v}C'D_v] = [-D_1, 0, \ldots, 0]. \tag{5.35}$$

An advantage of Theorem 5.1 over the method of Theorem 5.2 is, however, that ill-conditioned matrices will not occur due to eigenvalues of the system matrix close to the origin. Moreover, Theorem 5.1 is directly applicable to systems with singular system matrix A.

5.4 MINIMAL-ORDER MINIMUM-TIME STATE OBSERVERS

We have seen that since the system outputs (5.2) yield direct information on m linear combinations of the state vector, an observer need only be of dimension $n - m$ in order to reconstruct the remaining $n - m$ inaccessible state variables. Two procedures for minimal-order deadbeat observer design, both involving zero-value eigenvalue assignment, are considered and are based respectively on the results of Section 2.5 and Section 2.4.

In the first method, the system is assumed to be in the observable companion form (5.12) to (5.15). The ith subsystem, $i = 1, 2, \ldots, m$, described by the state vector

$$\bar{x}_i(k) = \begin{bmatrix} \bar{x}_{\sigma_{i-1}+1}(k) \\ \vdots \\ \bar{x}_{\sigma_i}(k) \end{bmatrix} \in R^{d_i} \tag{5.36}$$

satisfies

$$\bar{x}_i(k+1) = \bar{A}_{ii}\bar{x}_i(k) + \sum_{\substack{j=1\\ j\neq i}}^{m} \bar{A}_{ij}\bar{x}_j(k) + [M^{-1}B]_i u(k) \tag{5.37}$$

where $\bar{A}_{ii} \in R^{d_i \times d_i}$ and $\bar{A}_{ij} \in R^{d_i \times d_j}$ denote the leading diagonal and off-diagonal blocks of (5.14) respectively and $[M^{-1}B]_i \in R^{d_i \times r}$ is the ith submatrix of $M^{-1}B$. Also, similar to the continuous-time formulation (2.82), we may redefine the output vector as

$$\bar{y}(k) = \left[\begin{array}{ccccc|c|ccccc} 0 & 0 & \cdots & 0 & 1 & & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \bar{x}(k)$$

$$= \begin{bmatrix} \bar{x}_{\sigma_1}(k) \\ \bar{x}_{\sigma_2}(k) \\ \vdots \\ \bar{x}_{\sigma_m}(k) \end{bmatrix} \tag{5.38}$$

so that the $\bar{x}_{\sigma_i}$, $i = 1, 2, \ldots, m$, components of $\bar{x}(k)$ are available. Moreover, since the first $d_i - 1$ columns of $\bar{A}_{ij}$ are identically zero, the ith subsystem (5.37) is seen to be a d_i-dimensional single output companion system driven by the directly measurable system signals $\bar{x}_{\sigma_j}(k)$ $(j \neq i)$ and $u(k)$. Therefore, an observer corresponding to the continuous-time one of (2.68) and (2.69) can be employed for each of the m subsystems.

In order to appreciate this better, let us pause for a moment to consider the design of an $(n-1)$th order deadbeat observer for the single-output nth order system in companion form

$$x(k+1) = \begin{bmatrix} 0 & 0 & \cdots & & -a_0 \\ 1 & 0 & \cdots & & -a_1 \\ 0 & 1 & & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} x(k) + M^{-1}Bu(k) \tag{5.39}$$

$$y(k) = [0 \quad \cdots \quad 0 \quad 1]x(k). \tag{5.40}$$

Theorem 5.3 *The state vector $x(k) \in R^n$ of the discrete system (5.39) and (5.40) is reconstructed in at most $n-1$ steps by an observer of order $n-1$ of the form*

$$z(k+1) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} z(k) + \begin{bmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{n-2} \end{bmatrix} y(k) + TM^{-1}Bu(k) \tag{5.41}$$

$$\hat{x}(k) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & & 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} y(k) \tag{5.42}$$

$$T = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in R^{(n-1)\times n}. \tag{5.43}$$

Proof The proof is immediate upon noting that (5.41) to (5.43) is a special case of the discrete-time equivalent of the continuous-time observer (2.68) and (2.69), obtained by setting the coefficients of the characteristic equation β_i, $i = 0, 1, \ldots, n-2$, equal to zero. In particular, the observer error $\varepsilon(k) \triangleq z(k) - Tx(k)$ satisfies

$$\varepsilon(k+1) = D\varepsilon(k); \quad \varepsilon(0) = z(0) - Tx(0) \tag{5.44}$$

where

$$D = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} \tag{5.45}$$

is a nilpotent matrix with an index of nilpotency of $n-1$. Therefore, noting that $y(k) = x_n(k)$, we have from (5.41) to (5.43) that

$$z(n-1) = Tx(n-1) = [x_1(n-1) \cdots x_{n-1}(n-1)]' \tag{5.46}$$

Q.E.D. □

It is remarked that the eigenvalues of D, corresponding to the characteristic polynomial λ^{n-1} ($\beta_0 = \beta_1 = \cdots = \beta_{n-2} = 0$) are all of zero value (cf. Equation (2.62) of Section 2.5).

Returning now to the general problem, it is desired to construct a $(d_i - 1)$

dimensional observer for the ith subsystem described by (5.37), where the last component of the state vector $\bar{x}_i(k) \in R^{d_i}$ is available in the single output

$$\bar{y}_i(k) = [0 \quad 0 \quad \cdots \quad 0 \quad 1]\bar{x}_i(k). \tag{5.47}$$

By Theorem 5.3, the required $(d_i - 1)$ dimensional observer is of the form

$$z_i(k+1) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} z_i(k) + \begin{bmatrix} a_{i0} \\ a_{i1} \\ \vdots \\ a_{i(d_i-2)} \end{bmatrix} \bar{y}_i(k)$$

$$+ T_i \sum_{\substack{j=1 \\ j \neq i}}^{m} \hat{a}_{ij}\bar{y}_j(k) + T_i[M^{-1}B]_i u(k) \tag{5.48}$$

$$\hat{x}_i(k) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} z_i(k) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \bar{y}_i \tag{5.49}$$

$$T_i = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in R^{(d_i - 1) \times d_i} \tag{5.50}$$

where $\hat{a}_{ij} \in R^{d_i}$ is the right-most column vector of $\bar{A}_{ij}$ in (5.37). The state vector of the ith subsystem is reconstructed in at most $d_i - 1$ steps. Thus it can be said that $x(k)$ is reconstructed within max $\{d_i - 1\}$ steps by designing an observer (5.48)–(5.50) for each subsystem. In other words, $x(k)$ can be reconstructed within $\nu - 1$ steps, while the total dimension of the observers is $\sum_{i=1}^{m} (d_i - 1) = n - m$.

The second approach to minimal-order observer design is closely related to the first in that, in accordance with Section 2.3, the system (5.1) is transformed to a co-ordinate basis where the first m state variables are directly contained in the outputs or $C = [I_m, 0]$ in (5.2). A minimal-order observer which reconstructs the remaining $n - m$ state variables is described by (2.34) as

$$z(k+1) = Dz(k) + Ey(k) + Hu(k) \tag{5.51}$$

$$\hat{x}(k) = Pz(k) + Vy(k) \tag{5.52}$$

where D, E, H, P and V are matrix functions of the $(n - m) \times m$ gain matrix K. In particular, the associated observer error $\varepsilon(k) \in R^{n-m}$ propagates according to (1.96) as

$$\varepsilon(k + 1) = D\varepsilon(k); \quad D = A_{22} - KA_{12} \tag{5.53}$$

where A_{12} and A_{22} are appropriate partitions of A.

Thus the minimal-order minimum-state reconstruction problem reduces to one of choosing the $(n - m) \times m$ observer gain matrix K such that $\varepsilon(k) = 0$ in a minimum number of steps for any initial observer error $\varepsilon(0)$. With reference to Equations (5.5) to (5.7), the problem is analogous to the full-order state reconstruction problem of Section 5.2 in that the pair (A_{22}, A_{12}) is equivalent in form to the pair (A, C). Pursuing this analogy, Theorem 1.9 implies that the system state may be reconstructed by a minimal-order observer in a minimum number of steps equal to $\bar{\nu}$, the observability index of the pair (A_{22}, A_{12}). But, by Lemma 2.1 it is known that if (A, C) is a completely observable pair of observability index ν, then (A_{22}, A_{12}) is also an observable pair of observability index $\bar{\nu} = \nu - 1$. These observations issue in the following lemma.

Lemma 5.1 *The state vector of the system (5.1) and (5.2) may be reconstructed by a minimal-order observer (5.51) and (5.52) in a minimum number of steps equal to $\nu - 1$.*

Thus by (5.53), the minimal-order minimum-time observer design problem is one of finding an observer gain matrix K such that

$$\varepsilon(\nu - 1) = (A_{22} - KA_{12})^{\nu-1}\varepsilon(0) = 0 \tag{5.54}$$

for any initial observer error $\varepsilon(0)$. Noting the direct analogy between (5.6) and (5.54), explicit expressions for the observer gain matrix are furnished in the following two theorems by adducing Theorem 5.1 and Theorem 5.2. It is assumed that A_{12} is of full rank $m < n - m$.

Theorem 5.4 *If the discrete linear system (5.1) and (5.2) is ν-step observable in the sense of (1.38), the state vector is reconstructed exactly in $\nu - 1$ steps by a minimal-order observer (5.51) and (5.52) parameterized by the $(n - m) \times m$ gain matrix given by*

$$K = M_m\bar{A}_{22m}(\bar{A}_{12m})^{-1}. \tag{5.55}$$

Theorem 5.5 *If the discrete linear system (5.1) and (5.2) is ν-step state reconstructible in the sense of (5.8), the state vector is reconstructed exactly in $\nu - 1$ steps by a minimal-order observer (5.51) and (5.52) parameterized by the $(n - m) \times m$ gain matrix K satisfying*

$$K'[A'^{-1}_{22}A'_{12}D_1, \ldots, A'^{-\nu}_{22}A'_{12}D_\nu] = [-D_1, 0, \ldots, 0]. \tag{5.56}$$

5.5 DEADBEAT SYSTEM CONTROL WITH INACCESSIBLE STATE

Chapter 4 discussed the problem of dynamical observer-based controller design to overcome the problem of system control where only certain state variables are present in the system outputs. In the problem of deadbeat control, defined in Section 5.2 for systems with completely accessible state, we adopt the design philosophy of Chapter 4. That is, the design procedure is separated into two parts: first synthesize an observer to reconstruct the system state vector in a minimum number of steps; then design a deadbeat controller on the basis of linear feedback of the reconstructed state rather than the actual inaccessible state.

Consider the completely observable and controllable discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{5.57}$$

$$y(k) = Cx(k) \tag{5.58}$$

where the output matrix C is of full rank $m < n$.

Since the state $x(k) \in R^n$ is not completely available for feedback control purposes, the control strategy is assumed to be of the feedback form

$$u(k) = F\hat{x}(k) \tag{5.59}$$

where $\hat{x}(k) \in R^n$ is an estimate of the state $x(k)$ and is generated by the minimal-order observer

$$z(k+1) = \mathrm{D}z(k) + \mathrm{E}y(k) + \mathrm{H}u(k) \tag{5.60}$$

$$\hat{x}(k) = Pz(k) + Vy(k) \tag{5.61}$$

where the observer state vector $z(k) \in R^{n-m}$ and D, E, H, P and V are of compatible dimensions. Furthermore, the matrices in (5.60) and (5.61) are required to satisfy the basic observer relations of Chapter 1:

$$PT + VC = I_n \tag{5.62}$$

$$TA - DT = EC \tag{5.63}$$

$$H = TB. \tag{5.64}$$

Let the observer state error be

$$\varepsilon(k) = z(k) - Tx(k) \tag{5.65}$$

where for any arbitrary system initial state $x(0)$,

$$\varepsilon(0) = z(0) - Tx(0). \tag{5.66}$$

Then, by Equations (5.58), (5.61), (5.62) and (5.65), the state reconstruction error is given by

$$e(k) \triangleq \hat{x}(k) - x(k) = P\varepsilon(k) \tag{5.67}$$

while by Equations (5.57), (5.58), (5.60), (5.63), (5.64) and (5.65) the observer error propagates according to

$$\varepsilon(k+1) = D\varepsilon(k), \quad k = 0, 1, \ldots \tag{5.68}$$

Using (5.59) and (5.67), the feedback controller is given by

$$u(k) = Fx(k) + FP\varepsilon(k). \tag{5.69}$$

Hence, the overall closed-loop system response may be described, via Equations (5.57), (5.68) and (5.69) by

$$\begin{bmatrix} x(k+1) \\ \varepsilon(k+1) \end{bmatrix} = \begin{bmatrix} A + BF & BFP \\ 0 & D \end{bmatrix} \begin{bmatrix} x(k) \\ \varepsilon(k) \end{bmatrix}. \tag{5.70}$$

In contrast to the accessible deadbeat control problem defined in Section 5.2, the deadbeat control problem with inaccessible state may be defined as one of constructing an observer-based controller of the form (5.59) to (5.61) such that, for any initial state $x(0)$ and observer error $\varepsilon(0)$, the state vector of the augmented system (5.70) goes to the origin in a minimum number of time steps. In view of the block diagonal structure of (5.70), this problem may be divided and solved in two parts. First, design a minimal-order state observer to reconstruct the state of the completely observable (state reconstructible) system (5.57) and (5.58) in a minimum of steps equal to $v - 1$, according to Theorem 5.4 (Theorem 5.5). Then, solve the deadbeat control problem with *accessible* state which is one of choosing the feedback gain matrix F such that the system state is driven from any arbitrary initial state to the origin in a minimum number of steps.

As the dual of the observer state reconstruction problem, it is established in Section 5.2 that if v_c is the controllability index of the system (5.1), the minimum number of steps is equal to v_c. Thus, the deadbeat controller design problem is one of choosing F such that

$$(A + BF)^{v_c} = 0. \tag{5.71}$$

One solution, based on the system controllability criterion (5.9), is presented in Equation (5.30). A second solution is obtained through dualizing the procedure culminating in Theorem 5.1. Namely, assume as the dual of the observability criterion (1.38) that the system state is completely v_c-step reachable in the sense that

$$\operatorname{rank}\,[B\ AB \cdots A^{v_c - 1}B] = n \quad \text{for some } v_c \leqslant n. \tag{5.72}$$

Then, as the dual of Theorem 5.1 we have the following theorem.

Theorem 5.6 *If the discrete linear system (5.57) is v_c-step reachable in the sense of (5.72), then under the linear feedback control law*

$$u(k) = Fx(k), \quad k = 0, 1, 2, \ldots \tag{5.73}$$

where

$$F = -\bar{B}_r^{-1}\bar{A}_r M_c$$

the system initial state is transferred to the origin in at most v_c steps and Equation (5.71) is satisfied.

The $r \times n$ matrix $\bar{A}_r$ and the $r \times r$ upper triangular matrix $\bar{B}_r$ contain the non-trivial rows of the controllable companion form matrices $\bar{A}$ and $\bar{B}$ respectively. M_c, the dual of M in (5.12) to (5.15), is the associated state transformation matrix. As in the dual theorem, Theorem 5.1, it is not necessary to assume that the state transition matrix A is non-singular.

In view of Equation (5.70) and the attendant separation of the problem into a minimum-time state reconstruction problem and a deadbeat control problem, a complete solution to the inaccessible state deadbeat control problem is summarized in the following.

Theorem 5.7 *The inaccessible state vector $x(k)$ of the discrete system (5.57) and (5.58) is driven from any arbitrary initial state to the origin in at most $v_c + v - 1$ steps by an observer-based controller (5.59) to (5.61) of which the observer is specified by (5.48) to (5.50), Theorem 5.4 or Theorem 5.5 and the controller is determined by (5.30) or Theorem 5.6. In other words, after at most $v - 1$ steps perfect observer state reconstruction ensues, the control law (5.59) is quite the same as (5.34) and the linear regulator transfers $x(v - 1)$ to the origin in at most v_c further steps.*

5.6 MINIMUM-TIME LINEAR FUNCTION OBSERVERS

We have seen in Chapter 3 that rather than having to first reconstruct the complete state vector, reconstruction of a linear state function such as the control law (5.34) can be directly accomplished by a linear function observer often of state dimension substantially lower than that of a minimal-order state observer. The problem of minimum-time discrete linear state function reconstruction is to determine an observer

$$z(k+1) = Dz(k) + Ey(k) + Hu(k), \quad z \in R^p \tag{5.74}$$

$$w(k) = Mz(k) + Ny(k) \tag{5.75}$$

such that the output $w(k) \in R^p$ estimates a given linear function, say $Lx(k)$,

exactly after a finite number of steps π, i.e.

$$w(k) = Lx(k), \quad k \geqslant \pi \tag{5.76}$$

for some integer π starting at $k = 0$.

Corresponding to the continuous-time results of Theorem 3.2, necessary and sufficient conditions for the existence of such a minimum-time linear function observer are contained in the lemma.

Lemma 5.2 *The system (5.74) and (5.75) is a π-step linear function observer for the controllable linear discrete system (5.1) and (5.2) in the sense of (5.76) if and only if there exists a $p \times n$ matrix T such that the quintuple (T, D, E, M, N) satisfies*

$$MD^k(TA - DT - EC) = 0 \tag{5.77}$$

$$MD^{\pi+k} = 0 \quad \text{for each } k \geqslant 0 \tag{5.78}$$

$$MD^k(TB - H) = 0 \tag{5.79}$$

$$L = MT + NC. \tag{5.80}$$

Proof (*sufficiency*) Assume that (5.77)–(5.80) are satisfied for some T. From (5.1), (5.2) and (5.74), we have

$$w(k + \pi) - Lx(k + \pi) = MD^{\pi+k}(z(0) - Tx(0)) = 0 \tag{5.81}$$

for each $k \geqslant 0$.

Proof (*necessity*) Assume that (5.74) and (5.75) is a π-step linear function observer. By (5.76), (5.75) and (5.2),

$$Mz(\pi + k) = (L - NC)x(\pi + k)$$

for each $k \geqslant 0$. A straightforward algebraic manipulation yields

$$MD^i z(\pi) = R_i x(\pi) + \sum_{j=1}^{i} (R_{j-1}B - MD^{j-1}H)u(\pi + i - j)$$

where R_i is a matrix defined sequentially by

$$R_0 = L - NC, \quad R_{i+1} = R_i A - MD^i EC. \tag{5.82}$$

Since the $u(i)$ and $x(\pi)$ are arbitrary (note the controllability assumption on (5.1)), we conclude that

$$R_k B = MD^k H$$

and there exists a matrix T such that

$$MD^k T = R_k \tag{5.83}$$

for each k. These two relations imply (5.79). From (5.83) and (5.82), we have

$$MD^kTA = R_{k+1} + MD^kEC = MD^{k+1}T + MD^kEC.$$

This implies (5.77). Setting $k = 0$ in (5.83) gives (5.80). On the basis of (5.77), (5.79) and (5.80), we have

$$w(\pi + k) = Lx(\pi + k) + MD^{\pi+k}(z(0) - Tx(0)).$$

Therefore $MD^{\pi+k}(z(0) - Tx(0)) = 0$, for each k, $z(0)$ and $x(0)$. This implies (5.78). Q.E.D. □

It may be assumed, without loss of generality, that the pair (D, M) is completely observable, since otherwise a lower dimensional observer can be obtained by eliminating the unobservable states of (5.74) and (5.75). In this respect Equations (5.77), (5.78) and (5.80) become

$$TA - DT = EC, \quad L = MT + NC, \quad D^\pi = 0 \tag{5.84}$$

and (5.79) becomes

$$H = TB. \tag{5.85}$$

Once a T is found to satisfy the observer constraint Equations (5.84), the matrix H of (5.85) is readily obtained. Thus, in the satisfaction of the fundamental relations (5.84), the problem solution hinges on the requirement that the $p \times p$ observer coefficient matrix D be nilpotent with index of nilpotency π. As such, it has two main aspects: first, it is desired to attain a state function reconstruction in a minimum numbers of steps π; a second requirement is that the observer order p be a minimum. The former aspect is the subject of the remainder of this section while the latter aspect is discussed in [K24], [A3] from a geometric point of view.

Corresponding to Theorem 1.9 on complete state reconstructibility, we have the following necessary and sufficient condition for reconstruction of the state function $Lx(\cdot)$ in a finite number of steps N from knowledge of past and present inputs and outputs.

Theorem 5.8 *The linear state function $Lx(\cdot)$ of the system (5.1) and (5.2) is reconstructible if and only if there exists an integer N, $0 \leqslant N \leqslant n$, such that*

$$\mathscr{R}[A'^N L'] \subseteq \mathscr{R}[C' A'C' \cdots A'^N C'] \tag{5.86}$$

where $\mathscr{R}[\cdot]$ denotes the range space.

Proof (*sufficiency*) Assume that (5.86) is satisfied. Then there exist matrices $\beta_i \in R^{s \times m}$, $i = 0, 1, \ldots, N$, such that

$$LA^N = \beta_0 C + \beta_1 CA + \cdots + \beta_N CA^N. \tag{5.87}$$

Equation (5.87) and the fact that $x(k)$ can be expressed by iterating (5.1) as

$$\begin{aligned} x(k) = {} & A^N x(k-N) + A^{N-1}Bu(k-N) + A^{N-2}Bu(k-N+1) \\ & + \ldots + Bu(k-1) \end{aligned} \tag{5.88}$$

imply that

$$\begin{aligned} Lx(k) &= LA^N x(k-N) + LA^{N-1}Bu(k-N) + \ldots + LBu(k-1) \\ &= (\beta_0 C + \beta_1 CA + \ldots + \beta_N CA^N)x(k-N) + LA^{N-1}Bu(k-N) \\ &\quad + \ldots + LBu(k-1) \\ &= \beta_0 y(k-N) + \beta_1\{y(k-N+1) - CBu(k-N)\} \\ &\quad + \ldots + \beta_N\{y(k) - CA^{N-1}Bu(k-N) - \ldots - CBu(k-1)\} \\ &\quad + LA^{N-1}Bu(k-N) + \ldots + LBu(k-1). \end{aligned} \tag{5.89}$$

Thus, the value of Lx can be determined from the knowledge of the finite sequence of past and present inputs and outputs. *Necessity* of (5.86) is proved by contradiction for which the reader is referred to [N1].

Corollary 5.1 *The minimum number of steps π required to reconstruct the linear function $Lx(\cdot)$ is given by*

$$\pi = \min \{N; \mathscr{R}[A'^N L'] \subseteq \mathscr{R}[C' A'C' \cdots A'^N C']\}. \tag{5.90}$$

Comparing Theorem 5.8 with Theorem 1.9, it is clear that $\pi \leqslant \nu - 1$ where ν is the system observability index. That is to say, the minimum number of steps can be reduced through using a deadbeat linear function observer rather than a deadbeat state observer.

Restricting our attention to linear function observers of order $n - m$, in general greater than the minimal order attainable, consider a special class of observer (5.74) and (5.75) where

$$\begin{gathered} D = TAG_0, \quad E = TAH_0, \quad H = TB \\ M = LG_0 \quad \text{and} \quad N = LH_0 \end{gathered} \tag{5.91}$$

satisfy the constraint Equations (5.84) and (5.85) and T is of full rank $n - m$. Define $e(k)$ and $\varepsilon(k)$ by

$$e(k) \triangleq w(k) - Lx(k), \quad \varepsilon(k) \triangleq z(k) - Tx(k). \tag{5.92}$$

Then, from (5.81) and (5.91)

$$\begin{aligned} e(\pi) &= w(\pi) - Lx(\pi) \\ &= LG_0 D^\pi \varepsilon(0). \end{aligned} \tag{5.93}$$

Since T is of full rank, there is a vector $\alpha_0 \in R^n$ such that $\varepsilon(0) = T\alpha_0$. Then, by (5.91) and (5.80),

$$e(\pi) = LG_0D^\pi T\alpha_0 = LG_0T(AG_0T)^\pi\alpha_0$$

$$= LG_0T(A - \phi C)^\pi\alpha_0 \tag{5.94}$$

where

$$\phi \triangleq AH_0. \tag{5.95}$$

Thus, by (5.94), the problem of reconstructing $Lx(\cdot)$ in a minimum of π steps so as to satisfy $D^\pi = 0$ in (5.84) is resolved by choosing ϕ in (5.95) so that

$$(A - \phi C)^\pi = 0. \tag{5.96}$$

This problem is similar to that of (5.19) in Section 5.3, so that we may adduce the result of Theorem 5.1 to give Theorem 5.9 where all matrices $\tilde{M}$, $\bar{A}_m$ and $\bar{C}_m$ are correspondingly defined.

Theorem 5.9 *If the linear state function $Lx(\cdot)$ of the discrete system (5.1) and (5.2) is reconstructible in the sense of (5.86), it may be reconstructed exactly in π steps by an observer (5.74), (5.75) and (5.91) of order $n - m$ parameterized by the $n \times m$ gain matrix ϕ of (5.95) given by*

$$\phi = \tilde{M}\bar{A}_m\bar{C}_m^{-1}. \tag{5.97}$$

The above result is readily shown to be equivalent to the minimum time non-minimal-order one originally presented by Nagata *et al.* [N1]. That is, by (5.14) and (5.15)

$$\bar{C} = C\tilde{M} = \bar{C}_m\hat{C} \tag{5.98}$$

or

$$\bar{C}_m = C\tilde{M}\hat{C}' \tag{5.99}$$

and

$$\tilde{M}\bar{A}_m = \tilde{M}\bar{A}\hat{C}' = \tilde{M}(\tilde{M}^{-1}A\tilde{M})\hat{C}' = A\tilde{M}C'. \tag{5.100}$$

The matrices $\bar{A}_m$ and $\bar{C}_m$ consist of the non-trivial columns of $\bar{A}$ and $\bar{C}$ in (5.14) and (5.15) respectively and

$$\hat{C} = \left[\begin{array}{ccccccccc|c|ccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & & & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & & & 0 & \cdots & 0 & 1 \end{array}\right]. \tag{5.101}$$

Then, by (5.99) and (5.100), the $n \times m$ observer gain matrix ϕ in (5.97) becomes

$$\phi = AW_0(CW_0)^{-1} \tag{5.102}$$

where

$$W_0 \triangleq \tilde{M}\hat{C}'. \tag{5.103}$$

5.7 NOTES AND REFERENCES

The state-space minimum-time state reconstruction problem and the dual deadbeat control problem date from Kalman [K6]. For an overview of the subject and a discussion of its relation to other fundamental linear system concepts see O'Reilly [O20].

The first minimum-time state reconstruction problem formulation of Section 5.2 and its solution in Theorem 5.1 of Section 5.3 are based on Ichikawa [I2]. The original little known result is due to Ackermann [A2] while an equivalent result for the dual deadbeat control problem seems to have been independently derived by Prepelită [P20] and introduced to a wider audience in [P12]. A full-order minimum-time observer of O'Reilly [O12] is readily shown to be equivalent by pursuing an argument similar to that of Equations (5.98) to (5.103). Also, a corrected version of Ichikawa [I1] is shown by O'Reilly [O18] to be a special case of O'Reilly [O12] for $v = n/m$, an integer. The second minimum-time state reconstruction problem formulation, together with its solution in Theorem 5.2, is in the spirit of Kalman's *sui generis* contribution [K6], and follows the dual deadbeat controller derivation of O'Reilly [O20].

Of the two methods of minimal-order deadbeat observer design in Section 5.4, the first method, based on eigenvalue assignment for the observable companion form, is closest to Ichikawa [I2]. An equivalent treatment of the design is contained in Porter and Bradshaw [P13], while the practically unknown original contribution is due to Ackermann [A2]. The second method of minimal-order observer design culminating in Theorem 5.4 and Theorem 5.5 derives from Ichikawa [I3] and O'Reilly [O20] respectively. A minimal-order design of O'Reilly [O12] may also be shown to be equivalent to that of Theorem 5.4 for reasons similar to those discussed previously in the full-order observer construction.

Equivalent treatments of the inaccessible state deadbeat control problem are to be found in Porter and Bradshaw [P12], [P13], Ichikawa [I2] and O'Reilly [O17]. Theorem 5.6 is a general statement of the number of time-steps required by any one of these observer-based controllers to achieve deadbeat closed-loop system performance. Lemma 5.2 of the minimum-time linear function reconstruction problem is due to Kimura [K24]. Theorem 5.8 and Corollary 5.1 are an extension of the state reconstructibility concept of Weiss [W4], discussed in Chapter 1, to linear state function reconstructibility by Nagata *et al.* [N1]. The minimum-time linear function observer of Theorem 5.9 is equivalent to one originally developed by Nagata *et al.* [N1]. Minimum-time linear function observers that are *also* of minimal-order are explored in a geometric setting by Kimura [K24] and Akashi and Imai [A3]. The latter authors [A3] additionally treat linear function observers that are

insensitive to system parameter variations. Finally, a minimum-time state observer parameterized by a bounded gain matrix may be obtained by dualizing a recent deadbeat control solution of [F1].

Chapter 6

Observers and Linear Least-Squares Estimation for Stochastic Systems

6.1 INTRODUCTION

In previous chapters, it has been tacitly assumed that the linear state model, upon which the various observer designs are based, is accurately known. It is often the case, however, that uncertainties in the system description are by no means negligible and that needed measurements of various states are frequently accompanied by errors. In such instances, it may be more appropriate to use a stochastic model in which inherent uncertainties and measurement errors are described by random processes with known second-order statistics.

The present chapter considers the general linear least-squares estimation problem of estimating the state of both continuous-time and discrete-time linear time-varying stochastic systems, particularly when some but not all of the output measurements are noise-free. For such a problem the Kalman-Bucy filter [K7], [K17] of full system order, with a numerically ill-conditioned gain matrix due to a singular measurement noise covariance matrix, is inappropriate. Instead, a well-conditioned linear least-squares estimator of minimal order $n - m_1$ (n = system order, m_1 = number of noise-free measurements) is presented. Advantages of the minimal-order estimator include a reduction in computational requirements (particularly for time-invariant systems) and a mitigation of the effects of system modelling errors and non-linearities. Important special cases are when all the measurements are corrupted by additive white noise, in which case the optimal estimator reduces to the Kalman-Bucy filter [K7], [K17], and the "coloured" measurement noise solutions of Bucy [B22], Stear and Stubberud [S18] and Bryson and Henrikson [B20].

A concise derivation of the linear least-squares stochastic estimator is presented using the fundamental "whitening filter" or innovations approach,

developed by Kailath [K1], for continuous and discrete systems in Sections 6.2 and 6.3 respectively. It is observed that while the continuous-time solution results in a minimal-order estimator substantially different from the Kalman filter, the presence of noise-free measurements does not result in an estimator radically different from the Kalman filter in the discrete-time case.

In Section 6.4 it is shown how these linear least-squares estimators may be realized by stochastic observers in which, for the continuous-time solution, the need for actual differentiation of the noise-free measurements is removed. Section 6.5 treats the related problem of suboptimal or specific optimal observer-estimator design wherein the observer configuration (order) is fixed *a priori* and its design matrices are then chosen in some minimum mean-square error sense. Finally, in Section 6.6 the linear least-squares estimator is shown to satisfy a separation principle, similar to that met in Chapter 4, in which the state estimate generated by the estimator is used in the standard optimal linear regulator feedback control law as if it were the actual inaccessible state of the system.

6.2 THE CONTINUOUS LINEAR LEAST-SQUARES ESTIMATOR

Consider the linear stochastic system described by*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \xi(t) \tag{6.1}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \eta_2(t) \end{bmatrix} \tag{6.2}$$

where $x(t) \equiv [x_1'(t), x_2'(t)]' \in R^n$ is a state vector, $u(t) \in R^r$ is a known control vector and $y(t) \in R^m$ is a measurement vector. $A(t)$, $B(t)$ and $C(t)$ are continuous matrices of consistent dimensions and $C(t)$ is further assumed to be of full rank $m \leqslant n$. The system disturbance $\xi(t)$, measurement noise $\eta_2(t)$ (of dimension $m_2 = m - m_1$) and system initial state $x(t_0)$ are random vectors described by the second-order statistics:

$$E[x(t_0)] = \bar{x}_0; \quad E[(x(t_0) - \bar{x}_0)(x(t_0) - \bar{x}_0)'] = \Sigma^0 = \Sigma^{0\prime} \tag{6.3}$$

$$E[\xi(t)] = 0; \quad E[\xi(t)\xi'(\tau)] = Q(t)\,\delta(t - \tau)$$

$$E[\eta_2(t)] = 0; \quad E[\eta_2(t)\eta_2'(\tau)] = R_{22}(t)\,\delta(t - \tau)$$

$$E[\xi(t)\eta_2'(\tau)] = S(t)\,\delta(t - \tau)$$

---

* In this chapter, all differential equations of the form of (6.1), where $\xi(\cdot)$ is a vector white noise process, are to be taken as shorthand expressions for stochastic differential equations of the type

$$dx(t) = A(t)x(t)\,dt + B(t)u(t)\,dt + d\xi(t)$$

where $d\xi(t)$ is an appropriate vector Brownian motion process, and are to be interpreted in the sense of Ito [P2].

where $Q(t) = Q'(t) \geqslant 0$, $R_{22}(t) = R'_{22}(t) > 0$ and $S(t)$ is a known $n \times m_2$ matrix.

Of interest is the construction of the linear least-squares estimator of minimal order for the stochastic system (6.1) and (6.2). That is, it is required to generate an estimate $\hat{x}(t/\tau)$* of the state $x(t)$ as a linear function of the available measurements $y(\tau), t_0 \leqslant \tau \leqslant t$, such that the mean square error $\Sigma(t)$ defined by

$$\Sigma(t) = E[(\hat{x}(t) - x(t))(\hat{x}(t) - x(t))'] \tag{6.4}$$

is a minimum for all $t_0 \leqslant t \leqslant T$. The following theorem gives us this optimal minimal-order estimator.

Theorem 6.1 *The optimal minimal-order estimator† is given by*

$$\begin{aligned}\dot{\hat{x}}_2(t) &= A_{22}(t)\hat{x}_2(t) + A_{21}(t)\hat{x}_1(t) + B_2(t)u(t) \\ &\quad + K_1(t)[\dot{y}_1(t) - H(t)y_1(t) - M(t)\hat{x}_2(t) - C_1(t)B(t)u(t)] \\ &\quad + K_2(t)[y_2(t) - C_{21}(t)C_{11}(t)^{-1}y_1(t) - N(t)\hat{x}_2(t)]\end{aligned} \tag{6.5}$$

$$\hat{x}_1(t) = C_{11}^{-1}(t)(y_1(t) - C_{12}(t)\hat{x}_2(t)) \tag{6.6}$$

where, suppressing explicit time-dependence,

$$\begin{aligned}H &\triangleq (\dot{C}_{11} + C_{11}A_{11} + C_{12}A_{21})C_{11}^{-1} \\ M &\triangleq \dot{C}_{12} + C_{12}A_{22} + C_{11}A_{12} - HC_{12} \\ N &\triangleq C_{22} - C_{21}C_{11}^{-1}C_{12}\end{aligned}$$

and the gain matrix $K(t) = [K_1(t), K_2(t)]$ satisfies

$$K(t) = (\Sigma_{22}[M' \mid N'] + [(C_{11}Q_{12} + C_{12}Q_{22})' \mid S_2])\hat{R}^{-1} \tag{6.7}$$

$$\begin{aligned}\dot{\Sigma}_{22}(t) &= (A_{22} - A_{21}C_{11}^{-1}A_{21})\Sigma_{22}(t) \\ &\quad + \Sigma_{22}(t)(A_{22} - A_{21}C_{11}^{-1}A_{21})' - K\hat{R}K' + Q_{22}\end{aligned} \tag{6.8}$$

where

$$\hat{R}(t) \triangleq \begin{bmatrix} C_1QC'_1 & C_1S \\ S'C'_1 & R_{22} \end{bmatrix}.$$

The estimator initial conditions are specified by

$$\hat{x}_2(t_0) = \bar{x}_{20} + K_1(t_0)(y_1(t_0) - C_1\bar{x}_0)$$

---

* $\hat{x}(t)$ is a shorthand notation for $\hat{x}(t/t)$.

† $A(t)$ and $C(t)$ are appropriately partitioned as

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \quad \text{and} \quad C(t) = \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} \quad \text{etc.}$$

$$\Sigma_{22}(t_0) = \Sigma_{22}^0 - [\Sigma_{21}^0 \mid \Sigma_{22}^0]C_1'K_1'(t_0) - K_1(t_0)C_1\begin{bmatrix}\Sigma_{12}^0\\ \Sigma_{22}^0\end{bmatrix}$$
$$+ K_1(t_0)C_1\begin{bmatrix}\Sigma_{11}^0 & \Sigma_{12}^0\\ \Sigma_{21}^0 & \Sigma_{22}^0\end{bmatrix}C_1'K_1'(t_0) \tag{6.9}$$

where

$$K_1(t_0) = [\Sigma_{21}^0 \mid \Sigma_{22}^0]C_1'\left(C_1\begin{bmatrix}\Sigma_{11}^0 & \Sigma_{12}^0\\ \Sigma_{21}^0 & \Sigma_{22}^0\end{bmatrix}C_1'\right)^{-1}. \tag{6.10}$$

A concise derivation of Theorem 6.1 by an innovations approach is deferred to Section 6.2.1. The linear least-squares estimator is an optimal one among the class of *all* estimators (linear or non-linear) in that, if the random processes (6.3) are of Gaussian distribution, it will generate the conditional-mean state estimate. Moreover, such a conditional-mean state estimator is optimal for a variety of error criteria other than for a mean square error one. Conditions sufficient for the uniform asymptotic stability (U.A.S.) of the estimator are as follows.

Theorem 6.2 *Suppose the pair*

$$\left(A_{22} - [(C_{11}Q_{12} + C_{12}Q_{22})' \mid S_2]\hat{R}^{-1}\begin{bmatrix}M\\ N\end{bmatrix}, D\right)$$

is uniformly completely controllable, where

$$Q_{22} - [(C_{11}Q_{21} + C_{12}Q_{22})' \mid S_2]\hat{R}^{-1}[(C_{11}Q_{21} + C_{12}Q_{22})' \mid S_2]' \equiv DD'$$

and the pair $\left(A_{22}, \begin{bmatrix}M\\ N\end{bmatrix}\right)$ *is uniformly completely observable. Then the estimator is U.A.S.*

The proof of Theorem 6.2 based on a Lyapunov stability analysis of the Riccati Equation (6.8), due to Kalman [K9], is briefly outlined.

Proof It may be assumed, without loss of generality (Kalman [K9]), that

$$[(C_{11}Q_{12} + C_{12}Q_{22})', S_2] \equiv 0_{(n-m_1)\times m}. \tag{6.11}$$

Then, specifically, it is required to prove the uniform asymptotic stability of

$$\dot{\varepsilon}(t) = (A_{22}(t) - \Sigma_{22}(t)\bar{C}'(t)\hat{R}^{-1}(t)\bar{C}(t))\varepsilon(t) \tag{6.12}$$

for all $t \in [t_0, \infty]$, where

$$\bar{C}(t) \triangleq \begin{bmatrix}M(t)\\ N(t)\end{bmatrix}$$

and $\hat{R}(t)$ is as previously defined. To this purpose, define the Lyapunov function

$$V_L(\varepsilon(t), t) = \varepsilon'(t)\Sigma_{22}^{-1}(t)\varepsilon(t). \tag{6.13}$$

If $(A_{22}(t), D(t))$ is uniformly completely controllable $((A_{22}(t), \bar{C}(t))$ is uniformly completely observable) it is straightforward although tedious to show (Kalman [K9]) that $V_L(\varepsilon(t), t)$ has a uniform upper (lower) bound; i.e. there exist positive constants β_1 and β_2 such that

$$\beta_1 \varepsilon'(t)\varepsilon(t) \leqslant V_L(\varepsilon(t), t) \leqslant \beta_2 \varepsilon'(t)\varepsilon(t) \quad \text{for all } t \in [t_0, \infty]. \tag{6.14}$$

By (6.8), (6.11) to (6.13) and the identity

$$\dot{\Sigma}_{22}^{-1}(t) \equiv -\Sigma_{22}^{-1}(t)\dot{\Sigma}_{22}(t)\Sigma_{22}^{-1}(t) \tag{6.15}$$

noting that Σ_{22}^{-1} exists by the *a priori* bounds in (6.14),

$$\dot{V}_L(\varepsilon(t), t) = -\varepsilon(t)(\bar{C}'(t)\hat{R}^{-1}(t)\bar{C}(t) + \Sigma_{22}^{-1}(t)D(t)D'(t)\Sigma_{22}^{-1}(t))\varepsilon(t) \tag{6.16}$$

which is in general negative semi-definite and so $V_L(\varepsilon(t), t)$ is non-increasing along any solution path of (6.12). Therefore (6.14) and (6.16) are together sufficient to prove that (6.12) is (at least) uniformly stable in the sense of Lyapunov (Kalman and Bertram [K15]).

In order to establish that (6.12) is, moreover, uniformly asymptotically stable it suffices to note the strict inequality

$$V_L(\varepsilon(t+\tau), t+\tau) - V_L(\varepsilon(t), t) = \int_t^{t+\tau} \dot{V}_L(\varepsilon(\lambda), \lambda)\, d\lambda < 0 \tag{6.17}$$

for all $t \in [t_0, \infty]$, which follows from Equation (6.16) and the assumptions of uniform complete observability and controllability. Q.E.D. □

6.2.1 Derivation of the fundamental equations

The linear least-squares estimator of Theorem 6.1 is derived using the "whitening filter" or innovations approach of Kailath [K1]. First of all, recall that $y_1(t)$ in Equation (6.2) represents m_1 exact measurements of m_1 linearly independent combinations of the state vector $x(t)$. Since $C(t)$ in (6.2) is of full rank $m \geqslant m_1$, $C_{11}(t)$ may be assumed to be an $m_1 \times m_1$ non-singular matrix. Also, an estimator of dimension $n - m_1$ only is required in order to estimate the remaining $n - m_1$ combinations of the state vector corresponding to $x_2(t)$.

Now the noise-free measurements $y_1(t)$ provide, by differentiation, statistical information, in addition to that provided by $y_2(t)$, about the unknown part of the state vector $x_2(t)$. Statistical information about $x_2(t)$, equivalent to that of $\dot{y}_1(t)$ and $y_2(t)$, is contained in the measurement innovations

$$v_1(t) = \dot{y}_1(t) - \hat{\dot{y}}_1(t/t) = \dot{y}_1(t) - (\dot{C}_1 + C_1A)\hat{x}(t) - C_1Bu(t)$$
$$v_2(t) = y_2(t) - \hat{y}_2(t/t) = y_2(t) - C_2\hat{x}(t) \tag{6.18}$$

where it is assumed that $\dot{y}_1(t)$ contains non-singular white noise (i.e. $C_1QC_1' > 0$, $C_1 \equiv [C_{11}, C_{12}]$).

Thus a linear least-squares estimator for the unknown system state $x_2(t)$ of (6.1) is as follows:

$$\hat{x}_2(t) = \int_{t_0}^{t} [g_1(t,\tau), g_2(t,\tau)]\begin{bmatrix} v_1(\tau) \\ v_2(\tau) \end{bmatrix} d\tau \tag{6.19}$$

where the linear filter $g(t,\tau) \triangleq [g_1(t,\tau), g_2(t,\tau)]$ is chosen so as to safisfy the projection theorem

$$E\left[(\hat{x}_2(t) - x_2(t))\begin{bmatrix} v_1(\tau) \\ v_2(\tau) \end{bmatrix}'\right] = 0, \quad t_0 \leqslant \tau \leqslant t. \tag{6.20}$$

The projection theorem is the fundamental theorem of linear least-squares estimation and furnishes necessary and sufficient conditions for a best linear estimate. Denoting $[v_1'(t), v_2'(t)]'$ by $v(t)$,

$$E[v(t)v'(\tau)] = \hat{R}(t)\,\delta(t-\tau) \tag{6.21}$$

where

$$\hat{R}(t) = \begin{bmatrix} C_1QC_1' & C_1S \\ S'C_1' & R_{22} \end{bmatrix}. \tag{6.22}$$

By Equations (6.18) to (6.22)

$$\begin{aligned} E[x_2(t)v'(\tau)] &= \int_{t_0}^{t} g(t,s)E[v(s)v'(\tau)]\,ds \\ &= g(t,\tau)\hat{R}(\tau), \quad t_0 \leqslant \tau \leqslant t. \end{aligned} \tag{6.23}$$

Thus, by (6.19) and (6.23), we now have

$$\hat{x}_2(t) = \int_{t_0}^{t} E[x_2(t)v'(\tau)]\hat{R}^{-1}(\tau)v(\tau)\,d\tau \tag{6.24}$$

which upon differentiation and using (6.1), (6.2) and (6.19), yields

$$\dot{\hat{x}}_2(t) = A_{22}\hat{x}_2(t) + A_{21}\hat{x}_1(t) + B_2u(t) + E[x_2(t)v'(t)]\hat{R}^{-1}(t)v(t) \tag{6.25}$$

$$\hat{x}_1(t) = C_{11}^{-1}[y_1(t) - C_{12}\hat{x}_2(t)] \tag{6.26}$$

where it is assumed that

$$E[\xi(t)v'(\tau)] = 0, \quad \tau < t. \tag{6.27}$$

By (6.26)

$$\hat{x}(t) = \left[\frac{C_{11}^{-1} y_1}{0}\right] + \left[\frac{-C_{11}^{-1} C_{12}}{I_{n-m_1}}\right]\hat{x}_2 \tag{6.28}$$

so that (6.18) may be written as

$$\begin{aligned} v_1(t) &= \dot{y}_1(t) - Hy_1(t) - M\hat{x}_2(t) - C_1 Bu(t) \\ v_2(t) &= y_2(t) - C_{21}C_{11}^{-1}y_1(t) - N\hat{x}_2(t) \end{aligned} \tag{6.29}$$

where

$$\begin{aligned} H &\triangleq (\dot{C}_{11} + C_{11}A_{11} + C_{12}A_{21})C_{11}^{-1} \\ M &\triangleq \dot{C}_{12} + C_{12}A_{22} + C_{11}A_{12} - HC_{12} \\ N &\triangleq C_{22} - C_{21}C_{11}^{-1}C_{12}. \end{aligned} \tag{6.30}$$

Therefore, in Equations (6.25), (6.26) and (6.29) we have Equations (6.5) and (6.6) of Theorem 6.1 where the gain matrix $K(t)$ of (6.7) is defined by

$$K(t) = E[x_2(t)v'(t)]\hat{R}^{-1}(t). \tag{6.31}$$

Defining the state estimation error and its covariance by

$$\begin{aligned} e_2(t) &\triangleq x_2(t) - \hat{x}_2(t) \\ \Sigma_{22}(t) &\triangleq E[e_2(t)e_2'(t)] \end{aligned} \tag{6.32}$$

and using (6.1), (6.18) and (6.28)

$$\begin{aligned} E[x_2(t)v'(t)] &= E\left\{(\hat{x}_2 + e_2)e_2'\left(\left[\frac{C_{11}^{-1}C_{12}}{I_{n-m_1}}\right](\dot{C}_1 + C_1A)\right)' \right. \\ &\qquad \left. + (\hat{x}_2 + e_2)[\xi'C_1' \mid \eta_2']\right\} \\ &= E[e_2e_2']\left(\left[\frac{C_{11}^{-1}C_{12}}{I_{n-m_1}}\right](\dot{C}_1 + C_1A)\right)' + E[\hat{x}_2[\xi'C_1' \mid \eta_2']] \\ &= \Sigma_{22}[M' \mid N'] + [(C_{11}Q_{12} + C_{12}Q_{22})' \mid S_2]. \end{aligned} \tag{6.33}$$

Thus, substituting (6.33) in (6.31) one has Equation (6.7) of Theorem 6.1. Also, differentiation of (6.32) and the use of (6.1) and (6.5) readily yields Equation (6.8).

Given the initial state statistics, the estimator initial conditions (6.9) and (6.10) are a standard application of least-squares linear regression of $x_2(t_0)$ on the initial noise-free observations $y_1(t_0)$ [D7]. That is,

$$\hat{x}_2(t_0) = \bar{x}_{20} + XY^{-1}(y_1(t_0) - \bar{y}_{10}) \tag{6.34}$$

where

$$X \triangleq E[(x_2(t_0) - \bar{x}_{20})(y_1(t_0) - \bar{y}_{10})']$$
$$Y \triangleq E[(y_1(t_0) - \bar{y}_{10})(y_1(t_0) - \bar{y}_{10})']$$

from which easily follows (6.9) and (6.10).

6.2.2 A parametric class of estimators

In the preceding analysis no special assumptions have been made about the linear stochastic system (6.1) and (6.2) beyond that the observation matrix $C(t)$ be of full rank $m \leqslant n$. It is recalled, however, from Section 2.3 that with or without transformation, as the case may be, $C(t)$ in (6.2) can have the simpler structure

$$C(t) = \begin{bmatrix} I_{m_1} & 0 \\ C_{21}(t) & C_{22}(t) \end{bmatrix}, \quad m_1 \leqslant m \leqslant n \tag{6.35}$$

where (6.1) and (6.2) are partitioned accordingly as before.

In this case, one obtains as a corollary to Theorem 6.1 a structurally simpler estimator which we may describe as a member of the parametric class of linear least-squares estimators.

Corollary 6.1 *The optimal minimal-order parametric estimator is given by*

$$\begin{aligned}\dot{\hat{x}}_2(t) = {} & A_{22}(t)\hat{x}_2(t) + A_{21}y_1(t) + B_2(t)u(t) \\ & + K_1(t)[\dot{y}_1(t) - A_{11}(t)y_1(t) - A_{12}(t)\hat{x}_2(t) - B_1(t)u(t)] \\ & + K_2(t)[y_2(t) - C_{21}(t)y_1(t) - C_{22}(t)\hat{x}_2(t)]\end{aligned} \tag{6.36}$$

$$\hat{x}_1(t) = y_1(t) \tag{6.37}$$

where the gain matrix $K(t) \triangleq [K_1(t), K_2(t)]$ *satisfies*

$$K(t) = (\Sigma_{22}(t)[A'_{12}(t), C'_{22}(t)] + [Q_{21}(t), S_2(t)])\hat{R}^{-1}(t) \tag{6.38}$$

$$\dot{\Sigma}_{22}(t) = A_{22}(t)\Sigma_{22}(t) + \Sigma_{22}(t)A'_{22}(t) - K(t)\hat{R}(t)K'(t) + Q_{22}(t) \tag{6.39}$$

where

$$\hat{R}(t) \triangleq \begin{bmatrix} Q_{11}(t) & S_1(t) \\ S'_1(t) & R_{22}(t) \end{bmatrix}.$$

The estimator initial conditions are specified by

$$\begin{aligned}\hat{x}_2(t_0) &= \bar{x}_{20} + K_1(t_0)(y_1(t_0) - \bar{x}_{10}) \\ \Sigma_{22}(t_0) &= \Sigma^0_{22} - K_1(t_0)\Sigma^0_{12} - \Sigma^0_{21}K'_1(t_0) + K_1(t_0)\Sigma^0_{11}K'_1(t_0)\end{aligned} \tag{6.40}$$

where

$$K_1(t_0) = \Sigma_{21}^0 \Sigma_{11}^{0-1}. \tag{6.41}$$

A matching simplification is achieved through substitution of (6.35) into Theorem 6.3 to obtain stability conditions for the parametric class of linear least-squares estimators.

6.3 THE DISCRETE LINEAR LEAST-SQUARES ESTIMATOR

Consider the linear discrete-time stochastic system described by

$$x(k+1) = A(k)x(k) + B(k)u(k) + \xi(k), \quad k = 0, 1, \dots \tag{6.42}$$

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1(k) \\ C_2(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \eta(k) \end{bmatrix} \tag{6.43}$$

where $x(k) = [x_1'(k), x_2'(k)]' \in R^n$ is a state vector, $u(k) \in R^r$ is a known control vector, and $y(k) \in R^m$ is a measurement vector. $A(k)$, $B(k)$ and $C(k)$ are real element matrices of compatible dimensions, and $C(k)$ is further assumed to be of full rank $m \leqslant n$. The system disturbance $\xi(k)$, measurement noise $\eta(k)$ (of dimension $m_2 = m - m_1$) and system initial state $x(0)$ are random vectors described by the second-order statistics:

$$E[x(0)] = \bar{x}_0; \quad E[(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)'] = \Sigma_0 = \Sigma_0' \tag{6.44}$$

$$E[\xi(i)] = 0; \quad E[\xi(i)\xi'(j)] = Q(i)\,\delta_{ij} \tag{6.45}$$

$$E[\eta(i)] = 0; \quad E[\eta(i)\eta'(j)] = R_{22}(i)\,\delta_{ij} \tag{6.46}$$

$$E[x(0)\xi'(i)] = 0; \quad E[x(0)\eta'(i)] = 0; \quad E[\xi(i)\eta'(j)] = 0 \tag{6.47}$$

where

$$Q(i) = Q'(i) \geqslant 0, \quad R_{22}(i) = R_{22}'(i) > 0.$$

Of interest is the construction of the linear least-squares estimator of minimal order for the stochastic system (6.42) and (6.43). That is, it is required to generate the filtered estimate $\hat{x}(k/k)$ of the state $x(k)$ as a linear function of the available measurements $y(j)$, $j = 0, 1, \dots, k$, such that the mean-square error $\Sigma(k)$ defined by

$$\Sigma(k) = E[(\hat{x}(k/k) - x(k))(\hat{x}(k/k) - x(k))'] \tag{6.48}$$

is a minimum for all $k = 0, 1, \dots$ Since $y_1(k)$ in (6.43) represents m_1 exact measurements of m_1 linearly independent combinations of the state vector $x(k)$, it is only required to estimate the remaining $n - m_1$ linearly independent combinations of $x(k)$ given by

$$z(k) = D(k)x(k). \tag{6.49}$$

Then if we define

$$\begin{bmatrix} C_1(k) \\ D(k) \end{bmatrix}^{-1} = [M(k), N(k)] \tag{6.50}$$

we have by (6.43) and (6.49) that

$$x(k) = M(k)z(k) + N(k)y_1(k). \tag{6.51}$$

Thus, since $y_1(k)$ is known, the estimation problem reduces to one of estimating the $(n - m_1)$ vector $z(k)$ as a linear function of the available measurements $y(j)$, $j = 0, 1, \ldots, k$. The following theorem gives us this optimal minimal-order estimator.

Theorem 6.3 *The optimal minimal-order estimator is given by the following set of recursions:*

$$\begin{aligned} \hat{z}(k+1) = {} & [D(k+1) - G(k+1)C(k+1)] \\ & \times \hat{x}(k+1/k) + G(k+1)y(k+1) \end{aligned} \tag{6.52}$$

$$\hat{x}(k+1/k) = A(k)\hat{x}(k/k) + B(k)u(k) \tag{6.53}$$

$$\hat{x}(k/k) = M(k)\hat{z}(k) + N(k)y_1(k) \tag{6.54}$$

$$\begin{aligned} \hat{z}(0) &= [D(0) - G(0)C(0)]\hat{x}(0/-1) + G(0)y(0) \\ \hat{x}(0/-1) &= \bar{x}_0 \end{aligned} \tag{6.55}$$

where the gain matrix $G(k)$ *satisfies*

$$G(k) = D(k)\Sigma(k/k-1)C'(k)[C(k)\Sigma(k/k-1)C'(k) + R(k)]^{-1} \tag{6.56}$$

$$\Sigma(k+1/k) = A(k)M(k)\Sigma_z(k/k)M'(k)A'(k) + Q(k) \tag{6.57}$$

$$\Sigma_z(k/k) = [D(k) - G(k)C(k)]\Sigma(k/k-1)D'(k) \tag{6.58}$$

$$\Sigma(0/-1) = \Sigma_0. \tag{6.59}$$

Proof The proof of this theorem using the orthogonal projection theorem may be slightly simplified by pursuing an innovations approach developed by Kailath [K1]. That is, consider the statistical information about $z(k)$, equivalent to that of $y(k)$, contained in the measurement innovations.

$$\nu(k) = y(k) - \hat{y}(k/k-1) = y(k) - C(k)\hat{x}(k/k-1). \tag{6.60}$$

Then a linear least-squares estimator for the $(n - m_1)$ vector $z(k)$ of (6.49) is

$$\hat{z}(k) = \sum_{i=0}^{k} G(k, i)\nu(i) \tag{6.61}$$

where the gain matrices $G(k, i)$ are chosen so as to satisfy the projection theorem

$$E[(z(k) - \hat{z}(k))v'(j)] = 0, \quad 0 \leqslant j \leqslant k. \tag{6.62}$$

Substitution of (6.49) and (6.61) in (6.62) yields

$$D(k)E[x(k)v'(j)] = \sum_{i=0}^{k} G(k, i)E[v(i)v'(j)]. \tag{6.63}$$

Since the innovations sequence $v(\cdot)$ is white, it is readily shown [K1] that

$$E[x(k)v'(k)] = \Sigma(k/k - 1)C'(k) \tag{6.64}$$

and

$$E[v(k)v'(k)] = C(k)\Sigma(k/k - 1)C'(k) + R(k) \tag{6.65}$$

whence, by (6.63) we have (6.56) of the theorem. Also by (6.61),

$$\hat{z}(k + 1) = \sum_{i=0}^{k} G(k + 1, i)v(i) + G(k + 1)v(k + 1) \tag{6.66}$$

$$= \hat{z}(k + 1/k) + G(k + 1)v(k + 1) \tag{6.67}$$

whereby, using (6.42), (6.43), (6.49), (6.51) and (6.60), Equations (6.52) to (6.55) follow. It is straightforward to establish the remaining Equations (6.56) to (6.59) upon noting that

$$\Sigma_z(k/k) \triangleq E[(z(k) - \hat{z}(k))(z(k) - \hat{z}(k))'] \tag{6.68}$$

the $(n - m_1) \times (n - m_1)$ filtered error covariance of $\hat{z}(k)$. Q.E.D. □

The above theorem is essentially that of Brammer [B16] extended to include control inputs and rearranged in the more suggestive form (6.52)–(6.55) admitting the following physical interpretation. Equation (6.52) represents the construction of the filtered estimate $\hat{z}(k + 1)$ as a linear combination of the predicted state estimate $\hat{x}(k + 1/k)$ and the current measurement $y(k + 1)$ (or innovations $v(k + 1)$). Equation (6.53) depicts the construction of the predicted state estimate $\hat{x}(k + 1/k)$ from the current filtered estimate $\hat{x}(k/k)$ and control input $u(k)$ modelling (6.42). Also (6.54) represents the filtered state estimate $\hat{x}(k/k)$ as a linear combination of the filtered estimate $\hat{z}(k)$ and the noise-free measurements $y_1(k)$. The initialization of the estimator in (6.55) is simply effected by setting $k = -1$ in (6.52).

Theorem 6.3 may be generalized in the manner of its continuous-time counterpart, Theorem 6.1, so as to permit cross-correlation between the system disturbance $\xi(k)$ and measurement noise $\eta(k)$ in the sense that

$$E\left[\xi(i)\begin{bmatrix} 0 \\ \eta(j) \end{bmatrix}'\right] = S(i)\,\delta_{ij}, \quad S \in R^{n \times m}. \tag{6.69}$$

It is left as an exercise for the reader to establish that the only change is that the one-step prediction error covariance (6.57) is now given by

$$\Sigma(k+1/k) = A(k)M(k)\Sigma_z(k/k)M'(k)A'(k) + A(k)M(k)G(k)S'(k) + S(k)G'(k)M'(k)A'(k) + Q(k). \tag{6.70}$$

Consideration of a discrete-time parametric class of estimators after the fashion of Section 6.2.2, where $C(k)$ is defined analogously to that of (6.35), affords some simplification of the equations of Theorem 6.3.

Finally, we note that the non-singularity of $C(k)\Sigma(k/k-1)C'(k) + R(k)$ ensures the existence and uniqueness of the optimal estimator as specified by (6.56). By relaxing the assumption of positive definiteness on $C(k)\Sigma(k/k-1)C'(k) + R(k)$, (6.56) may be appropriately generalized by use of a generalized inverse [B15]. We shall not pursue this refinement here, but instead note that the matrix is positive definite if, say, the $m_1 \times m_1$ matrix C_1QC_1' is positive definite and $R_{22}(k)$ is positive definite by assumption.

6.4 THE OPTIMAL STOCHASTIC OBSERVER-ESTIMATOR

In the previous two sections, a linear least-squares estimation theory for stochastic linear systems with singular measurement noise covariance was developed by a direct innovations approach without reference to observer theory. It is now shown how the structure of a minimal-order observer, with appropriate choice of parameters, may alternatively be used to realize the linear least-squares estimators of Theorem 6.1 and Theorem 6.3.

6.4.1 A continuous-time stochastic observer-estimator

Assuming, as in Section 6.2, that $\dot{y}_1(t)$ contains non-singular white noise (i.e. $C_1QC_1' > 0$), the minimal-order observer (1.27), (1.28), (1.39), (1.40) of Section 1.4 can form the basis of the following optimal observer-estimator.

Theorem 6.4 *The minimal-order observer-estimator* $(\hat{z}(t) \in R^{n-m_1})$, *described by*

$$\dot{\hat{z}}(t) = (TAP + \dot{T}P)\hat{z}(t) + (TAV + \dot{T}V)y_1(t) + TBu(t) + K_2(t)(y_2(t) - C_2\hat{x}(t)) \tag{6.71}$$

$$\hat{x}(t) = P(t)\hat{z}(t) + V(t)y_1(t) \tag{6.72}$$

where

$$T(t) = [0, I_{n-m_1}] - K_1(t)C_1(t) \tag{6.73}$$

$$P(t) = \left[\frac{-C_{11}^{-1}(t)C_{12}(t)}{I_{n-m_1}}\right], \quad V(t) = \left[\frac{C_{11}^{-1}(t)(I - C_{12}(t)K_1(t))}{K_1(t)}\right] \tag{6.74}$$

and the gain matrix $K(t) \triangleq [K_1(t), K_2(t)]$ is given by (6.7), is a linear least-squares estimator for the system (6.1) and (6.2). Moreover, the optimal observer initial condition is given by

$$z(t_0) = T(t_0)\bar{x}_0 \tag{6.75}$$

where $K_1(t_0)$ is given by (6.10).

Proof By (6.72) and (6.74), we have

$$\hat{x}_2(t) = \hat{z}(t) + K_1(t)y_1(t) \tag{6.76}$$

and Equation (6.6) of Theorem 6.1. Differentiation of (6.76) gives

$$\dot{\hat{x}}_2(t) = \dot{\hat{z}}(t) + \dot{K}_1(t)y_1(t) + K_1(t)\dot{y}_1(t) \tag{6.77}$$

whereupon substitution of (6.71) in (6.77) together with the use of (6.2), (6.73) and (6.74) yields Equation (6.5) of Theorem 6.1. The initial conditions (6.9) are obtained by substituting (6.75) and (6.73) in (6.72) and setting $t = t_0$. Q.E.D.

It is to be noticed that through the use of (6.76) the optimal observer-estimator of Theorem 6.4 does not require $\dot{y}_1(\cdot)$ as an input. Although by no means peculiar to the observer approach, this is a desirable feature from the point of view of implementation in that differentiators are avoided. Also, Equation (6.76) rewritten as

$$\hat{z}(t) = [-K_1(t), I_{n-m_1}]\begin{bmatrix} y_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{6.78}$$

is reminiscent of the deterministic observer relation (2.31) encountered in Chapter 2. In the stochastic state estimation situation, the significance of $K_1(t)$ is that it weights the innovations due to $\dot{y}_1(t)$ (while $K_2(t)$ weights the innovations due to $y_2(t)$), a fact that is explicitly exploited in the derivation of Theorem 6.1. A simplification of Theorem 6.4 is effected in the manner of Corollary 6.1, by considering a parametric class of observer-estimators in which the observation matrix $C(t)$ is given by (6.35).

6.4.2 A discrete-time stochastic observer-estimator

Similarly, the discrete-time minimal-order observer of Section 1.5 can be used to form the basis of an optimal minimal-order observer-estimator. By (6.43), (6.49) and (6.50), it is assumed without loss of generality that

$$z(k) = D(k)x(k), \quad D(k) = [0, I_{n-m_1}] \tag{6.79}$$

$$x(k) = \begin{bmatrix} -C_{11}^{-1}(k)C_{12}(k) \\ I_{n-m_1} \end{bmatrix} z(k) + \begin{bmatrix} C_{11}^{-1}(k) \\ 0 \end{bmatrix} y_1(k). \tag{6.80}$$

Theorem 6.5 *The minimal-order observer-estimator* $(\bar{p}(k) \in R^{n-m_1})$ *described by*

$$\begin{aligned} \hat{p}(k+1) = {} & T(k+1)A(k)P(k)\hat{p}(k) + T(k+1)A(k)V(k)y(k) \\ & + T(k+1)B(k)u(k) \end{aligned} \tag{6.81}$$

$$\hat{x}(k) = P(k)\hat{p}(k) + V(k)y(k) \tag{6.82}$$

where

$$T(k) = [0, I_{n-m_1}] - G(k)C(k) \tag{6.83}$$

$$P(k) = \begin{bmatrix} -C_{11}^{-1}(k)C_{12}(k) \\ \hdashline I_{n-m_1} \end{bmatrix}, \quad V(k) = \begin{bmatrix} C_{11}^{-1}(k)\{[I_{m_1}, 0] - C_{12}(k)G(k)\} \\ \hdashline G(k) \end{bmatrix} \tag{6.84}$$

and the gain matrix $G(k)$ *is given by (6.56) and (6.79), is a linear least-squares estimator for the system (6.42) and (6.43). Moreover, the optimal observer initial condition is given by*

$$\hat{p}(0) = T(0)\hat{x}(0/-1) = T(0)\bar{x}_0. \tag{6.85}$$

Proof By (6.81) and (6.82)

$$\hat{p}(k+1) = T(k+1)\hat{x}(k+1/k) \tag{6.86}$$

where $\hat{x}(k+1/k)$ is the *a priori* state estimate defined in (6.53). Also, using (6.82) and (6.84), we have

$$\hat{x}_2(k+1) = \hat{p}(k+1) + G(k+1)y(k+1). \tag{6.87}$$

Substitution of (6.86) in (6.87) and use of (6.83) results in the linear least-squares estimator

$$\hat{x}_2(k+1/k+1) = \hat{x}_2(k+1/k) + G(k+1)[y(k+1) - C(k+1)\hat{x}(k+1/k)] \tag{6.88}$$

which by (6.79), noting that $z(k) \equiv x_2(k)$, is essentially (6.52). The remaining equations of Theorem 6.3 follow accordingly. Q.E.D. □

Again, a further simplification of Theorem 6.5 is afforded by considering a parametric class of observer-estimators in which the observation matrix $C(k)$ is defined analogously to that of (6.35).

6.5 SUB-OPTIMAL STOCHASTIC OBSERVER-ESTIMATORS

A common feature of the optimal estimators previously discussed is that their order is reduced from full system order by the number of perfect (noise-free) system measurements, $y_1(\cdot)$. In other words, one requires the estimator to be optimal in a linear least-squares sense and then seeks the lowest or minimal-order realization.

On the other hand, if these so-called "noise-free measurements" $y_1(\cdot)$ are in fact slightly corrupted by noise, an acceptable, albeit sub-optimal, estimator of order $n - m_1$, $(0 \leqslant m_1 \leqslant m)$ may be designed. In this approach the estimator is constrained to be of a given dynamical order $l = n - m_1$, subject to which the estimator design parameter matrices are chosen in some minimum mean-square error sense.

6.5.1 A sub-optimal continuous-time observer

Consider the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + M(t)v(t) + \xi(t) \tag{6.89}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} I_{m_1} & 0 \\ C_{21}(t) & C_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix} v(t) + \begin{bmatrix} 0 \\ \eta_2(t) \end{bmatrix} \tag{6.90}$$

which is as in (6.1) and (6.2) except that $v(\cdot) \in R^q$ is a coloured noise (Markov one) process which satisfies

$$\dot{v}(t) = \phi(t)v(t) + n(t); \quad E[v(0)] = 0 \tag{6.91}$$

$$E[v(0)[v'(0), x'(0)]] = [P_{vv}(0), P_{vx}(0)]$$

$$E[n(t)] = 0; \quad E[n(t)n'(\tau)] = N(t)\,\delta(t - \tau).$$

It is to be observed that all the measurements (6.90) are corrupted by possibly different coloured noise processes. The first m_1 measurements are assumed to be of "good quality" relative to the remaining $m_2 = m - m_1$ components. As in (6.72) of the continuous-time optimal problem, addition of white noise into the measurements $y_1(t)$ is inadmissible because it would lead to an infinite state error variance.

Eliminating $x_1(t)$ in favour of $y_1(t)$ and $v(t)$, we have from (6.90) that

$$x_1(t) = y_1(t) - L_1(t)v(t) \tag{6.92}$$

and

$$\dot{x}_1(t) = \dot{y}_1(t) - \dot{L}_1(t)v(t) - L_1(t)\dot{v}(t). \tag{6.93}$$

Thus, by (6.89) to (6.91), one obtains

$$\begin{bmatrix} \dot{v}(t) \\ \dot{x}_2(t) \end{bmatrix} = \left[\begin{array}{c|c} \phi & 0 \\ \hline M_2 - A_{21}L_1 & A_{22} \end{array}\right] \begin{bmatrix} v(t) \\ x_2(t) \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline B_2u(t) + A_{21}y_1(t) \end{array}\right] + \begin{bmatrix} n(t) \\ \xi_2(t) \end{bmatrix} \tag{6.94}$$

and

$$\begin{aligned} z = \left[\begin{array}{c} z_1 \\ \hline z_2 \end{array}\right] &= \left[\begin{array}{c} \dot{y}_1 - A_{11}y_1 - B_1u \\ \hline y_2 - C_{21}y_1 \end{array}\right] \\ &= \left[\begin{array}{c} \dot{L}_1 - A_{11}L_1 + L_1\phi + M_{11}A_{12} \\ \hline L_2 - C_{21}L_1 \end{array}\right] \left[\begin{array}{c} v \\ \hline x_2 \end{array}\right] + \left[\begin{array}{c} L_1n + \xi_1 \\ \hline \eta_2 \end{array}\right] \end{aligned} \tag{6.95}$$

omitting explicit time-dependence.

The true linear least-squares estimator of Theorem 6.1 is directly applicable to the system (6.94) and (6.95). It will be of dimension $n + q - m_1$, and estimates the coloured noise state $v(t)$ as well as the substate $x_2(t)$.

As an alternative suboptimal approach, bearing in mind that $v(t)$ is a zero mean process, we may take the estimate of $v(t)$ as

$$\hat{v}(t) = E[v(t)] \equiv 0, \quad t \geqslant 0. \tag{6.96}$$

Then, by (6.94) to (6.96), a specific optimal observer-estimator for $x_2(\cdot) \in R^{n-m_1}$ takes the form

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y_1 + B_2u + K\left(z - \begin{bmatrix} A_{12} \\ C_{22} \end{bmatrix}\hat{x}_2\right). \tag{6.97}$$

We now seek a gain matrix K which is optimal in the sense of minimizing the weighted estimation error covariance.

$$J = E\{e'(t)\tilde{W}(t)e(t)\}, \quad e \triangleq x - \hat{x}. \tag{6.98}$$

Using (6.92), we have that

$$\begin{aligned} e_1 \triangleq x_1 - \hat{x}_1 &= -L_1(v - \hat{v}) \\ &\triangleq -L_1e_v \end{aligned} \tag{6.99}$$

since $\hat{v} = 0$. Then, with $W(t)$ defined as

$$W(t) = \begin{bmatrix} -L_1(t) & 0 \\ 0 & I \end{bmatrix}' \tilde{W}(t) \begin{bmatrix} -L_1(t) & 0 \\ 0 & I \end{bmatrix} \tag{6.100}$$

the cost (6.98) can be written as

$$J = \operatorname{tr}\{W(t)\Sigma(t)\} \tag{6.101}$$

in which

$$\Sigma(t) = E\left\{\begin{bmatrix} e_v \\ e_2 \end{bmatrix}[e_v', e_2']\right\}. \tag{6.102}$$

The minimization of (6.101) with respect to the gain $K(t)$ can be formulated [L4] as the terminal time optimization problem of finding a matrix $K(t)$, $t_0 \leqslant t \leqslant T$ so as to minimize the terminal time cost

$$J(T) = \operatorname{tr}[W(T)\Sigma(T)] \tag{6.103}$$

subject to the dynamic constraint

$$\dot{\Sigma}(t) = \Psi(K(t), t)\Sigma(t) + \Sigma(t)\Psi'(K(t), t) + \Gamma(K(t), t) \tag{6.104}$$

with given initial condition $\Sigma(t_0) = \Sigma_0$ where

$$\Psi(K(t), t) \triangleq \left[\begin{array}{c|c} \phi & 0 \\ \hline M_2 - A_{21}L_1 - KC_1 & A_{22} - KC_2 \end{array}\right] \tag{6.105}$$

and $\Gamma(K(t), t)$ is the covariance kernal of the white noise defined by

$$E\left\{\begin{bmatrix} n(t) \\ \xi_2(t) - K(t)v(t) \end{bmatrix}\begin{bmatrix} n(\tau) \\ \xi_2(\tau) - K(\tau)v(\tau) \end{bmatrix}'\right\} = \Gamma(K(t), t)\,\delta(t-\tau) \tag{6.106}$$

and

$$v(t) \triangleq \begin{bmatrix} L_1 n + \xi_1 \\ \eta_2 \end{bmatrix}. \tag{6.107}$$

The solution of the optimization problem is summarized in the following theorem which we state without proof (see [L4]).

Theorem 6.6 *The specific optimal observer-estimator (6.97) is parameterized by the $(n - m_1) \times m$ gain matrix*

$$K(t) = [W_{22}^{-1}(t)W_{21}(t) \quad I_{n-m_1}][\Sigma(t)\bar{C}(t) + S(t)] \times \left[\begin{array}{c|c} Q_{11}(t) + L_1(t)N(t)L_1'(t) & 0 \\ \hline 0 & R_{22}(t) \end{array}\right]^{-1} \tag{6.108}$$

where

$$\dot{\Sigma} = \Psi\Sigma + \Sigma\Psi' + \Gamma, \quad \Sigma(t_0) = \Sigma_0 \tag{6.109}$$

$$\bar{C}(t) \triangleq \left[\begin{array}{c|c} \dot{L}_1 - A_{11}L_1 + L_1\emptyset + M_1 & A_{12} \\ \hline L_2 - C_{21}L_1 & C_{22} \end{array}\right] \tag{6.110}$$

and

$$S(t) \triangleq \begin{bmatrix} NL_1' & 0 \\ Q_{12}' & 0 \end{bmatrix}. \tag{6.111}$$

Some points of comparison of Theorem 6.6 with the true linear least-squares estimators of Sections 6.2 and 6.4 are worthy of mention.

(i) The $(n - m_1) \times m$ observer gain K is related to the $(n + q - m_1) \times m$ least-square error gain

$$K^* = \begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix}$$

by

$$K = [W_{22}^{-1} W_{21}, I]K^*.$$

(ii) If the coloured noise processes are absent from (6.89) and (6.90) ($M = 0$, $L = 0$), the $(n - m_1)$th order specific optimal estimator of Theorem 6.6 reduces to the linear least-squares one of Corollary 6.1 and is independent of the weighting matrix W.

(iii) The observer gain (6.108) is not unique but depends on the choice of weighting matrix W.

(iv) There is no minimization of errors in $\hat{x}_1$ since, by (6.99), $e_1 = -L_1 v$ and cannot be affected by the observer.

6.5.2 A sub-optimal discrete-time observer

Unlike the specific optimal continuous-time observer, the discrete-time optimal observer-estimator of prespecified order $n - m_1$ is of the same structure as its linear least-squares counterpart of Theorem 6.5. As in the linear least-squares solution, direct feedthrough of noisy measurements into the state estimate (6.82) is permitted since the discrete white noise (6.46) is of finite variance (cf. continuous-time problem).

The observer-estimator is, however, sub-optimal since no filtering of the substate $x_1(\cdot) \in R^{m_1}$ is allowed. That is, by (6.82) and (6.84), one has

$$\hat{x}_1(k) = C_{11}^{-1}(k)[y_1(k) - C_{12}(k)\hat{x}_2(k)] \tag{6.112}$$

in which the estimate $\hat{x}_1(k)$ is degraded by the fact that $y_1(k)$ is no longer perfect but contains an unfiltered additive white noise component $\eta_1(k)$.

6.6 THE OPTIMAL STOCHASTIC CONTROL PROBLEM

The relevance of observer-estimator theory to extending the solution of the linear optimal stochastic control problem, otherwise known as the linear quadratic Gaussian (LQG) problem, to the general singular case is now examined.

Consider the linear stochastic system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + \xi(t) \\ y(t) &= C(t)x(t) + \eta(t) \end{aligned} \tag{6.113}$$

where $x(t_0)$, $\xi(t)$ and $\eta(t)$ are Gaussian random vectors with covariances Σ_0, $Q(t)$ and $R(t)$ respectively.

The set of *admissible* controls can only consist of those controls which depend on past measurement data; i.e. for $u(t)$ to be an admissible control, it must be of the form

$$u(t) = \psi(t, y(\tau)), \quad \tau \in [t_0, t] \tag{6.114}$$

where $\psi(t, \cdot)$ satisfies an additional Lipschitz condition which guarantees the existence of $x(t)$ in (6.113).

The performance index of the system is the quadratic one

$$J = E\left[x'(T)L_3x(T)\right] + \int_{t_0}^{T} (x'(t)L_1(t)x(t) + u'(t)L_2(t)u(t))\, dt\Big] \tag{6.115}$$

where $L_1 \geqslant 0$, $L_2 > 0$ and $L_3 \geqslant 0$ and the expectation is taken over all underlying uncertainties.

The optimal (stochastic) control problem may be stated as that of finding an admissible control of the form (6.114) such that the expected cost (6.115) is minimized subject to the dynamic constraint (6.113).

The solution to the above problem is summarized in the following theorem which we state without proof.

Theorem 6.7 (Separation Theorem) *The optimal admissible $u^*(t)$, which minimizes the expected cost (6.115) subject to (6.113) is given by*

$$u^*(t) = -L_2^{-1}(t)B'(t)N(t)\hat{x}(t) \tag{6.116}$$

where

$$\begin{aligned} \dot{N}(t) = &-N(t)A(t) - A'(t)N(t) - L_1(t) \\ &+ N(t)B(t)L_2^{-1}(t)B'(t)N(t), \quad N(T) = L_3 \end{aligned} \tag{6.117}$$

and the conditional mean state estimate

$$\hat{x}(t) \triangleq E[x(t) \mid Y(t)], \quad Y(t) = \{y(\sigma), \sigma \in [t_0, t]\}. \tag{6.118}$$

Proof See Wonham [W14] or Tse [T4]. □

Theorem 6.7, also known as the certainty equivalence theorem, states that the overall optimal control policy can be separated into two distinct parts:

(a) construct the conditional mean state estimate $\hat{x}(t)$;

(b) construct the optimal feedback control $u^*(t)$ as if the conditional mean state estimate $\hat{x}(t)$ were the actual (accessible) state of the system.

It is normally assumed that the conditional mean state estimate $\hat{x}(t)$ is provided by the Kalman-Bucy filter. However, all that is really required is that the conditional mean process has the same dynamics as the original state process except with different driving disturbance. Therefore, we might expect the estimator of Theorem 6.1 or Corollary 6.1 to provide such a process for the case where some of the measurements are noise-free.

To see this, we have, by Equations (6.36) and (6.37), that the complete state estimate $\hat{x}(t)$ satisfies

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \begin{bmatrix} I_{m_1} \\ K_1(t) \end{bmatrix}(\dot{y}(t) - A_{11}y_1(t) - A_{12}\hat{x}_2(t) - B_1u(t)) + \begin{bmatrix} 0 \\ K_2(t) \end{bmatrix}(y_2(t) - C_{21}x_1(t) - C_{22}\hat{x}_2(t)) \tag{6.119}$$

where $u(t)$ is an admissible control and explicit time-dependence of matrices is suppressed for notational convenience. It has already been observed that $(\dot{y}_1 - A_{11}y_1 - A_{12}\hat{x}_2 - B_1u)$ and $(y_2 - C_{21}x_1 - C_{22}\hat{x}_2)$ are white noise processes and so the proof of Theorem 6.7 goes through as before (Wonham [W14], Tse [T4]). Note that if, as has been mostly assumed throughout this chapter, the Gaussian assumption on the second-order random vectors x_0, $\xi(t)$ and $\eta_2(t)$ in Equations (6.1) and (6.2) is relaxed, $u^*(t)$ in (6.116) is the optimal linear control law.

6.7 NOTES AND REFERENCES

The original and in many ways, the most complete solution to the continuous-time noise-free measurements or singular stochastic state estimation problem of Section 6.2 is given by Bryson and Johansen [B21]. More recent work on the continuous-time problem stemmed from the desire to extend the deterministic observer theory of state reconstruction to stochastic systems [T7], [O25], [U1]. By exploiting more fully the noise-free system measurements, O'Reilly and Newmann [O25], [O3] were able to develop a structurally simpler minimal-order observer-estimator, found to be equivalent to the original Bryson-Johansen filter [B21].

The linear least-squares estimator of Theorem 6.1 is based on the innovations approach of [O10], a reference which contains additional discussion and clarification of earlier related work. Earlier equivalent results using Wiener-Hopf formulation are also to be found in [S13]. The stability result of Theorem 6.2 is very much in the spirit of Kalman and Bucy [K17],

and is proved in a way similar to the development of Kalman [K9]. Corollary 6.1 was first presented in [O25], while Theorem 6.4 is based on [U1] and [O25]. The continuous-time specific optimal observer design problem was originally considered by Newmann [N7]. The sub-optimal observer-estimator of Theorem 6.6 is patterned on the development of Leondes and Yocum [L4] for an equivalent structurally simpler system realization.

Historically, the development of a singular state estimation theory for discrete-time stochastic systems has proceeded along analogous lines to that of the continuous-time problem. A thorough discussion is given in [O15]. Theorem 6.3 is essentially that of Brammer [B16], the original and somewhat overlooked investigator, and is derived in Section 6.3 by the innovations approach of [K1]. The equivalent optimal observer-estimator of Theorem 6.5 is based on Yoshikawa and Kobayashi [Y3], Yoshikawa [Y1] and O'Reilly and Newmann [O27]. A discrete-time stability result corresponding to Theorem 6.2 is as yet unavailable but the recent results of Moore and Anderson [M17] hold promise for extension to the singular case. The original discrete-time specific optimal observer-estimator solution is due to Aoki and Huddle [A10]. Discrete-time sub-optimal reduced-order observer-estimators in which redundancy is removed through considering an equivalent class of systems are presented by Leondes and Novak [L2], [L3], [N9] and, equivalently, by Iglehart and Leondes [I5]. These results together with further discussion of the relation of the sub-optimal (almost singular) solution to the optimal (singular) one are contained in the review article [N10].

Section 6.6 discusses a simple though important extension of the non-singular linear quadratic Gaussian (*LQG*) solution to the singular case. The former solution, embodied in the renowned Separation Theorem 6.7 is most often attributed to Wonham [W14], and is treated in the many textbooks on optimal stochastic control (see also the special *LQG* issue [A14]). The discrete-time version of the separation theorem, appropriate to the estimators of Theorem 6.3 and Theorem 6.5, is entirely analogous to Theorem 6.7 and the main treatments are to be found in Joseph and Tou [J7] and [A14].

Throughout the chapter it has been assumed that a stochastic state–space model is known. Related recursive least squares formulae are derived and, indeed, further light is thrown on the differences between continuous and discrete-time problems through consideration of state–space models which generate a specified output signal covariance [KS]. Among other things, Gevers and Kailath [G2] have demonstrated that, unlike the continuous-time case, the discrete-time signal covariance need not contain a delta function (equivalent to white noise).

A frequency domain characterization of the linear least-squares estimator of Theorem 6.1 and Corollary 6.1 is considered in Chapter 8. There, the feedback nature of the estimator is emphasized in a derivation of the optimal return-

difference relation associated with a frequency domain description of the matrix Riccati equation. A related study of the asymptotic properties of a filter as the measurement noise covariance matrix approaches the null matrix was initiated by Friedland [F9] and is further developed by Kwakernaak and Sivan [K42], Shaked and Bobrovsky [S8] and O'Reilly [O28]. In particular, Kwakernaak and Sivan [K42] show that the state estimation error can be reduced to zero by reducing the measurement noise to zero if and only if (1) the number of measured variables is at least as large as the number of disturbing variables and (2) the zeros of the system are all in the left-half complex plane. The importance of the second condition for high-gain feedback systems is discussed in Section 8.4; see also [K41]. The finite-time stationary Wiener-Kalman type filter of Grimble [G7], [G8] may also be profitably extended to the general singular problem [O23].

While the optimal linear regulator of Theorem 4.2 enjoys impressive robustness properties, the *LQG* controller of Theorem 6.7 may possess unsatisfactory stability margins [D9]. Methods of overcoming this undesirable lack of robustness are discussed in Section 8.6; see also Doyle and Stein [D10] and Lehtomaki *et al.* [L1].

Finally, it should be borne in mind that our excursion into the realms of linear least-squares estimation theory has been a brief one; for general background, modern developments and connections with other fields, the exhaustive survey of Kailath [K2] will repay much study. Other texts in optimal linear filtering that provide useful supplementary information include the recent ones of Anderson and Moore [A9] and Kailath [K3].

Chapter 7

Adaptive Observers

7.1 INTRODUCTION

The various observers of previous chapters allow reconstruction of the state of a linear system from measurements of its inputs and outputs, provided the system parameters are known. Where *a priori* knowledge of the system parameters is lacking, asymptotic state reconstruction may still be achieved by deploying an *adaptive observer*. An adaptive observer is basically an observer in which both the parameters and state variables of the system are estimated simultaneously. Its importance stems not only from allowing adaptive system state reconstruction, but is also due to the fact that it constitutes a key component in the adaptive feedback control of unknown linear systems with inaccessible state. In both problems the foremost aim is that the overall non-linear adaptive scheme be globally asymptotically stable.

Broadly speaking, the several seemingly different types of adaptive observer appearing in the literature may be divided into two classes: *explicit* (model reference) adaptive observers and *implicit* (parameterized) observers.

The former class of adaptive observer is derived in Section 7.2 by means of a Lyapunov synthesis technique. A feature of this observer is that, in addition to parameter adaptation, certain auxiliary signals are fed into the observer to ensure global asymptotic convergence of the state reconstruction process. On the basis of a non-minimal realization of the unknown system in Section 7.3, a considerable simplification of the adaptive observer is achieved in that these auxiliary signals are no longer required. Section 7.4 discusses the non-trivial extension of the adaptive observer to multi-output systems.

Section 7.5 treats the design of the second class of adaptive observers, the class of implicit adaptive observers, in which the state reconstruction process is well separated from the parameter adaptation process. The role of the implicit adaptive observer in adaptive feedback control of unknown linear systems with inaccessible state is examined in Section 7.6. A most useful aspect of this adaptive observer is that its stability does not presume the system input and

output signals to be bounded, which is precisely what is required to be proven in any proof of stability (signal boundedness) of the overall closed-loop control system.

7.2 AN ADAPTIVE OBSERVER FOR A MINIMAL REALIZATION OF THE UNKNOWN SYSTEM

Consider the completely observable single-input single-output linear system described by

$$\begin{aligned} \dot{x} &= [-a \,|\, \bar{A}]x + bu \\ y &= c'x = x_1 \end{aligned} \tag{7.1}$$

where the constant column vectors a and b represent the *unknown* parameters of the system, $\bar{A} \in R^{n \times (n-1)}$ is a known matrix and $c' = [1, 0, \ldots, 0]$. It is observed from Section 2.5 that no loss of generality is incurred by assuming this state representation, otherwise known as a *minimal realization* where the number of unknown system parameters is a minimum. Special case structures of the known $\bar{A}$ matrix frequently encountered are

$$\bar{A} = \begin{bmatrix} I_{n-1} \\ 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 1 & \cdots & 1 \\ -\lambda_2 & & \\ & \ddots & \\ & & -\lambda_n \end{bmatrix}. \tag{7.2}$$

It is desired to construct a full-order observer that will simultaneously estimate the system state x and the unknown parameter vectors a and b. To this end, the system (7.1) may be rewritten as

$$\begin{aligned} \dot{x} &= Kx + [k - a]x_1 + bu \\ y &= c'x = x_1 \end{aligned} \tag{7.3}$$

where $K \triangleq [-k \,|\, \bar{A}]$ is a stable matrix. Notice that the unknown system parameters a and b are associated with the known signals $x_1(t)$ and $u(t)$. The structure of (7.3) suggests that the following system is a candidate for an adaptive full-order observer:

$$\dot{\hat{x}} = K\hat{x} + [k - \hat{a}(t)]x_1(t) + \hat{b}(t)u(t) + w^1(t) + w^2(t) \tag{7.4}$$

where $\hat{a}(t)$ and $\hat{b}(t)$ are estimates of the parameter vectors a and b, $\hat{x}(t)$ is an estimate of $x(t)$ and $w^1(t)$ and $w^2(t)$ are auxiliary n-dimensional signals required to stabilize the adaptive observer. The success of the adaptive scheme hinges on a suitable adjustment of the parameter estimates $\hat{a}$ and $\hat{b}$ such that if

$$\alpha(t) \triangleq a - \hat{a}(t), \quad \beta(t) \triangleq \hat{b}(t) - b, \quad e(t) \triangleq \hat{x}(t) - x(t) \tag{7.5}$$

$$\lim_{t\to\infty} \alpha(t) = 0; \quad \lim_{t\to\infty} \beta(t) = 0; \quad \lim_{t\to\infty} e(t) = 0. \tag{7.6}$$

Now, by (7.3), (7.4) and (7.5), the state estimation error $e(t)$ satisfies

$$\dot{e} = Ke + \theta z + w, \quad e_1 = h'e \tag{7.7}$$

where $\theta = [\alpha \,|\, \beta]$, $z' = [x_1, u]$ and $w = w^1 + w^2$.

7.2.1 Adaptive laws and proof of stability

Our objective is to determine from a stability analysis of (7.7) the signals w^1 and w^2 and the adaptive laws for $\dot{\theta}(t)$ in order to ensure asymptotic convergence in the sense of (7.6).

Lemma 7.1 *Given an arbitrary bounded function of time $z(t)$, there exist vectors $v(t)$ and $w(t)$ with $v(t) = G(p)z(t)$ ($p = d/dt$) and $w = w(\dot{\theta}, v)$ such that*

$$\dot{e} = Ke + \theta z + w, \quad e_1 = h'e \tag{7.8}$$

$$\dot{\varepsilon} = K\varepsilon + d\theta' v, \qquad \varepsilon_1 = h'\varepsilon \tag{7.9}$$

are equivalent in terms of the input(z) — output(e_1) representation provided the pair (K, h') is completely observable.

Proof Without loss of generality, assume that the pair (K, h') is in the observable companion form

$$K = \left[-k \left| \begin{matrix} I \\ 0 \end{matrix} \right. \right] \quad \text{and} \quad h' = [1 \quad 0 \quad \cdots \quad 0]. \tag{7.10}$$

Equivalence of (7.8) and (7.9) implies that

$$h'(pI - K)^{-1}[\theta z + w - d\theta' v] = 0$$

or

$$\sum_{i=1}^{n} p^{n-i}[\theta_i z + w_i - d_i \theta' v] = 0 \tag{7.11}$$

where the subscript i denotes the ith element of the corresponding vector.

By associating $\{w_i\}$ with terms containing only v and the first derivative of θ, and equating the coefficients of θ to zero, the following choice of d, v and w satisfies (7.11):

$$d' = [1 \quad d_2 \quad d_3 \quad \cdots \quad d_n] \tag{7.12}$$

$$v_i = \left[\frac{p^{n-i}}{p^{n-1} + d_2 p^{n-2} + \cdots + d_n}\right] z, \quad i = 1, 2, \ldots, n \tag{7.13}$$

$$w' = [0, \dot{\theta}' A_2 v, \dot{\theta}' A_3 v, \ldots, \dot{\theta}' A_n v] \tag{7.14}$$

$$A_m = \begin{array}{c} \begin{array}{ccccccc|cccc} & & & \overbrace{\hphantom{-d_{m+1}\ \cdot\ \cdot\ \cdot\ -d_{n-1}}}^{n-m+1} & & & & & \overbrace{\hphantom{0\ \cdot\ \cdot\ 0}}^{m-2} & & \end{array} \\ \left[\begin{array}{ccccccccccc} 0 & -d_m & -d_{m+1} & \cdot & \cdot & \cdot & -d_n & 0 & \cdot & \cdot & 0 \\ 0 & 0 & -d_m & \cdot & \cdot & \cdot & -d_{n-1} & -d_n & 0 & \cdot & 0 \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & -d_m & \cdot & \cdot & \cdot & \cdot & \cdot & -d_n \\ 0 & 1 & d_2 & \cdot & \cdot & \cdot & d_{m-1} & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1 & d_2 & \cdot & \cdot & d_{m-2} & d_{m-1} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 & d_2 & d_3 & \cdot & \cdot & d_{m-1} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\ \cdot\\0\end{array}}\right\} m-1 \\ \left.\vphantom{\begin{array}{c}0\\0\\ \cdot\\0\end{array}}\right\} n-m+1 \end{array} \end{array}$$

$$m = 2, 3, \ldots, n. \tag{7.15}$$

Hence with this choice of v and w, (7.8) and (7.9) have the same input–output relation. Q.E.D. □

The proof of Lemma 7.1 is a constructive one in that it provides expressions for the auxiliary signals v and the stabilizing feedback signals w^1 and w^2.

Special case: Particularly simple expressions for v and w arise when the triple $\{h', K, d\}$ is specified by

$$d' = h' = [1 \quad 0 \quad \cdots \quad 0]$$
$$K = \begin{bmatrix} -\lambda_1 & l' \\ 0 & \Lambda \end{bmatrix} \tag{7.16}$$

where the $(n-1) \times 1$ vector

$$l = [1 \quad 1 \quad \cdots \quad 1]' \tag{7.17}$$

and

$$\Lambda = \operatorname{diag}(-\lambda_2, -\lambda_3, \ldots, -\lambda_n) \tag{7.18}$$

and $\lambda_i > 0$, $\lambda_i \neq \lambda_j$ when $i \neq j$, for all $i, j \in \{1, 2, \ldots, n\}$.

The corresponding v and w are

$$v_i = \left[\frac{1}{p + \lambda_i}\right] z, \quad i = 2, 3, \ldots, n$$
$$v_1 = z \tag{7.19}$$

and

$$w' = [0, \dot{\theta}_2 v_2, \dot{\theta}_3 v_3, \ldots, \dot{\theta}_n v_n] \tag{7.20}$$

where $\{\theta_i\}$ are the elements of the vector θ.

By Lemma 7.1, we may study the stability of (7.9) instead of that of (7.8) to arrive at the following theorem.

Theorem 7.1 *Given a dynamical system represented by the completely observable and completely controllable triple $\{h', K, d\}$, where K is a stable matrix, a positive definite matrix $\Gamma = \Gamma' > 0$, and a vector $v(t)$ of arbitrary bounded functions of time. Then the system of differential equations*

$$\dot{\varepsilon} = K\varepsilon + d\theta' v, \quad e_1 = h'\varepsilon \tag{7.21a}$$

$$\dot{\theta} = -\Gamma e_1 v \tag{7.21b}$$

is stable provided that $H(s) \triangleq h'(sI - K)^{-1} d$ is strictly positive real. If furthermore, the components of the vector $v(t)$ are sinusoidal signals with at least n distinct frequencies, the system (7.21) is uniformly asymptotically stable.*

Proof Consider the following quadratic function as a candidate Lyapunov function for the system (7.21):

$$V = \tfrac{1}{2}[\varepsilon' P \varepsilon + \theta' \Gamma^{-1} \theta]. \tag{7.22}$$

Differentiation of (7.22) with respect to time, and substitution of (7.21) yields

$$\begin{aligned} \dot{V} &= \tfrac{1}{2}\varepsilon'(K'P + PK)\varepsilon + \varepsilon' P\, d\theta' v + \theta' \Gamma^{-1} \dot{\theta} \\ &= \tfrac{1}{2}\varepsilon'(K'P + PK)\varepsilon + \varepsilon'(Pd - h)\theta' v. \end{aligned} \tag{7.23}$$

If $H(s)$ is strictly positive real, then by the Kalman-Yakubovich lemma (Narendra and Kudva [N3]), there exist a vector g, matrices $P = P' > 0$ and $L = L' > 0$, and a sufficiently small scalar $\mu > 0$ such that

$$\begin{aligned} K'P + PK &= -gg' - \mu L \\ Pd &= h. \end{aligned} \tag{7.24}$$

Hence $\dot{V}$ in (7.23) is negative semidefinite and the system (7.21) is stable. Going on to consider motions along which $\dot{V} \equiv 0$ or $\varepsilon(t)$ is identically zero, it is seen from (7.21) that

$$\theta' v \equiv 0. \tag{7.25}$$

It can be shown (Kudva and Narendra [K39]) that if the input $u(t)$ has at least

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* A rational function $H(s)$ of the complex variable $s = \sigma + i\omega$ is positive real if (i) $H(\sigma)$ is real and (ii) Re $H(s) > 0$, for all $\sigma > 0$ and Re $H(s) \geqslant 0$ for $\sigma = 0$; the rational function $H(s)$ is strictly positive real if $H(s - \varepsilon)$ is positive real for some $\varepsilon > 0$.

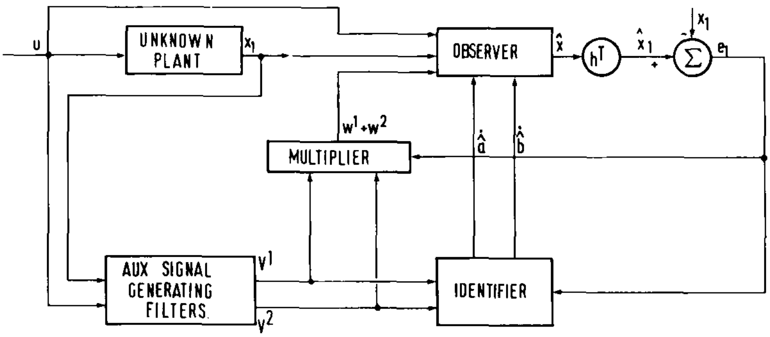

Fig. 7.1 Observer scheme for cases 1 and 2 for single-input single-output plant.

n distinct frequencies, the $2n$ elements of v are linearly independent with the consequence that the only solution of (7.25) is the trivial solution $\theta \triangleq [\alpha \mid \beta] = 0$. Hence, $e(t) = 0$ and the system (7.21) is uniformly asymptotically stable. Q.E.D. □

7.2.2 Final structure of the adaptive observer

In view of the preceding discussion, particularly Lemma 7.1 and Theorem 7.1, it is seen that the implementation of an adaptive observer (7.4) to reconstruct the system state according to the asymptotic design requirements of (7.5) involves the following:

(i) a suitable choice of system and observer companion form such as that of (7.10) or (7.16);
(ii) generation of the auxiliary signals $v^1(t)$ and $v^2(t)$ in (7.13) or (7.19) to form the adaptive laws (7.21b);
(iii) determination of the stabilizing feedback signals $w^1(t)$ and $w^2(t)$ in (7.14) or (7.20).

A general schematic of the adaptive observer is depicted in Fig. 7.1. Depending on which particular comrade representation (7.10) or (7.16) is used, the corresponding auxiliary signals v^1, v^2, and the feedback stabilizing signals w^1 and w^2 are displayed in Table 7.1 as Cases 1 and 2 respectively. In contrast to Case 1, it is noted in Case 2 that the system representation chosen gives rise to feedback signals w^1 and w^2 of simple structure. Although it is impossible to eliminate completely the signals w^1 and w^2 for any minimal realization of the system, these signals may be dispensed with when we presently study systems in non-minimal realization.

Table 7.1

Case	Canonical form	Auxiliary signals v^1 and v^2	Stabilizing feedback signals w^1 and w^2	Positive real condition
(1)	$\bar{A} = \begin{bmatrix} I \\ \hline 0 \end{bmatrix}$	$v_i^1 = \left[\dfrac{p^{n-i}}{p^{n-1} + d_2 p^{n-2} + \cdots + d_n}\right] x_1$	$w^1 = [0, \dot{\theta}^{1T} A_2 v^1, \ldots, \dot{\theta}^{1T} A_n v^1]^T$	$h^T(sI - K)^{-1} d =$
		$v_i^2 = \left[\dfrac{p^{n-i}}{p^{n-1} + d_2 p^{n-2} + \cdots + d_n}\right] u$	$w^2 = [0, \dot{\theta}^{2T} A_2 v^2, \ldots, \dot{\theta}^{2T} A_n v^2]^T$	$\dfrac{s^{n-1} + d_2 s^{n-2} + \cdots + d_n}{s^n + k_1 s^{n-1} + \cdots + k_n}$
		$i \in \{1, 2, 3, \ldots, n\}$	with A_m as defined in Equation (7.15)	strictly positive real
(2)	$\bar{A} = \begin{bmatrix} l^T \\ \hline \Lambda \end{bmatrix}$	$v_i^1 = \left[\dfrac{1}{p + \lambda_i}\right] x_1$	$w^1 = [0, \dot{\theta}_2^1 v_2^1, \ldots, \dot{\theta}_n^1 v_n^1]^T$	$h^T(sI - K)^{-1} d = \dfrac{l}{s + \lambda_1}$
	$k = (-\lambda_1, 0, \ldots, 0)^T$	$v_i^2 = \left[\dfrac{1}{p + \lambda_i}\right] u$	$w^2 = [0, \dot{\theta}_2^2 v_2^2, \ldots, \dot{\theta}_n^2 v_n^2]^T$	strictly positive real
		$i \in \{2, 3, \ldots, n\}$		
		$v_1^1 = x_1$; $v_1^2 = u$		

7.3 AN ADAPTIVE OBSERVER FOR A NON-MINIMAL REALIZATION OF THE UNKNOWN SYSTEM

Consider the minimal realization of an unknown system (7.1) and (7.2), namely

$$\dot{v} = \left[-a \left| \frac{l'}{\Lambda} \right. \right] v + bu$$
$$y = c'v = v_1 \tag{7.26}$$

where $l \in R^{n-1}$ and $\Lambda \in R^{(n-1)\times(n-1)}$ are defined by (7.17) and (7.18). Defining the additional state vectors $z_1(t) \in R^{n-1}$ and $z_2(t) \in R^{n-1}$ as solutions of the state equations

$$\dot{z}_1 = \Lambda z_1 + ly, \quad z_1(0) = 0$$
$$\dot{z}_2 = \Lambda z_2 + lu, \quad z_2(0) = z_{20}$$

we have by (7.26) and the above equations that the non-minimal state–space representation is given by

$$\dot{x} = \begin{bmatrix} -a_1 & -\bar{a}' & \bar{b}' \\ l & \Lambda & 0 \\ 0 & 0 & \Lambda \end{bmatrix} x + \begin{bmatrix} b_1 \\ 0 \\ l \end{bmatrix} u, \quad x(0) = \begin{bmatrix} y(0) \\ 0 \\ z_{20} \end{bmatrix} \tag{7.27}$$
$$y = c'x$$

where

$$a \triangleq \begin{bmatrix} a_1 \\ \bar{a} \end{bmatrix} \quad \text{and} \quad b \triangleq \begin{bmatrix} b_1 \\ \bar{b} \end{bmatrix}$$

and

$$x \triangleq [y, z_1', z_2']' \in R^{2n-1}.$$

A full-order adaptive observer for the non-minimal system realization (7.27) is

$$\dot{\hat{x}} = \begin{bmatrix} 0 & -\hat{\bar{a}}' & \hat{\bar{b}}' \\ l & \Lambda & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \hat{x} + \begin{bmatrix} \hat{b}_1 \\ 0 \\ l \end{bmatrix} u - \lambda_1(\hat{y} - y) - \hat{a}_1 y \tag{7.28}$$
$$\hat{x}(0) = 0, \quad \lambda_1 > 0$$

where, as usual, the caret denotes estimated parameters and states.

The state estimation error $e(t) = \hat{x}(t) - x(t) \triangleq [e_y, e_{z_1}', e_{z_2}']'$ of which from (7.27) and (7.28) and the initial conditions,

$$e_{z_1} \equiv 0. \tag{7.29}$$

If, further, the parameter error vectors are defined as

$$\begin{aligned}\alpha' &= (\alpha_1, \alpha_2') \triangleq [a_1 - \hat{a}_1, (\bar{a} - \hat{\bar{a}})'] \\ \beta' &= (\beta_1, \beta_2') \triangleq [\hat{b}_1 - b_1, (\hat{\bar{b}} - \bar{b})']\end{aligned} \tag{7.30}$$

the error equations become

$$\begin{aligned}\dot{e}_y &= -\lambda_1 e_y + \alpha_1 y + \alpha_2' \hat{z}_1 + \beta_1 u + \beta_2' \hat{z}_2 + \bar{b}' e_{z_2} \\ \dot{e}_{z_2} &= \Lambda e_{z_2}.\end{aligned} \tag{7.31}$$

Finally, upon defining the error vectors

$$\varepsilon \triangleq (e_y, e_{z_2}')', \quad \theta \triangleq (\alpha, \beta) \tag{7.32}$$

we have

$$\dot{\varepsilon} = K\varepsilon + d\theta' v, \quad e_y = h'\varepsilon \tag{7.33}$$

where

$$\begin{aligned}k &= \begin{bmatrix} -\lambda_1 & \bar{b}' \\ 0 & \Lambda \end{bmatrix} \\ d &= h = [1, 0, \ldots, 0]' \\ v &= [v^1, v^2] = [(y, \hat{z}_1')', (u, \hat{z}_2')'].\end{aligned} \tag{7.34}$$

Since Equation (7.33) is similar in form to (7.21a) and the transfer function $h'(sI - K)^{-1} = 1/(s + \lambda_1)$ is strictly positive real, it follows from Theorem 7.1 that the adaptive laws

$$\begin{aligned}\dot{\alpha} &= -\Gamma_1 e_y v^1, \quad \Gamma_1 = \Gamma_1' > 0 \\ \dot{\beta} &= -\Gamma_2 e_y v^2, \quad \Gamma_2 = \Gamma_2' > 0\end{aligned} \tag{7.35}$$

are stable.

As in the minimal realization case, given a stable observer coefficient matrix K, the speed of convergence of the overall scheme is determined by the magnitude and frequency content of the inputs and the choice of adaptive gains Γ_1 and Γ_2 in (7.35). The system zero initial condition on z_1 in (7.27) is a technical restriction made so as to eliminate e_{z_1} from (7.31) and (7.33). Notice from (7.34) and Fig. 7.2 that, in contrast to the minimal-realization case, the auxiliary signals v^1 and v^2 are generated within the $(2n - 1)$-dimensional observer and that no stabilizing feedback signals are required.

7.3.1 State estimation

Recall that our earlier objective was the estimation of the state vector $v(t)$ of the minimal-realization system (7.26). While it is possible to estimate the minimal-realization system state $v \in R^n$ by a suitable transformation of the

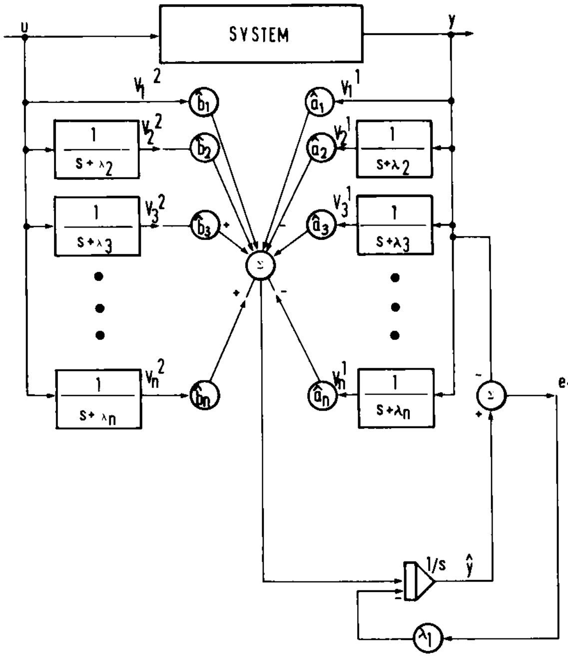

Fig. 7.2 Non-minimal realization observer.

non-minimal-realization observer state vector $\hat{x} \in R^{2n-1}$, an alternative method is as follows.

By using $\hat{a}(t)$ and $\hat{b}(t)$ in lieu of a and b, an observer for the system (7.26) may be constructed as

$$\dot{\hat{v}} = \left[-\hat{a}(t) - g(t) \left| \frac{l'}{\Lambda} \right. \right] v + g(t)y + \hat{b}(t)u. \tag{7.36}$$

The gain vector $g(t)$ is determined from the relation $g(t) = -\hat{a}(t) - k$, where the vector k is such that

$$\left[k \left| \frac{l'}{\Lambda} \right. \right]$$

becomes a desired stable observer coefficient matrix. That is, if $p(s) \triangleq s^n + d_1 s^{n-1} + \cdots + d_n$ is a desired asymptotically stable characteristic polynomial, then $k \triangleq [k_1, k_2, \ldots, k_n]'$ is determined from the relation

$$s - k_1 - \frac{k_2}{s + \lambda_2} - \cdots - \frac{k_n}{s + \lambda_n} = \frac{p(s)}{(s + \lambda_2)(s + \lambda_3) \cdots (s + \lambda_n)}. \tag{7.37}$$

In other words,

$$\begin{aligned} k_1 &= -d_1 + \sum_{i=2}^{n} \lambda_i \\ k_i &= -\left[p(s)(s+\lambda_i) \Big/ \prod_{i=2}^{n} (s+\lambda_i) \right]_{s=-\lambda_i}, \quad i = 2, 3, \ldots, n \quad . \end{aligned} \tag{7.38}$$

Since the vector k can be predetermined from $p(s)$ and λ_i, $i = 2, 3, \ldots, n$, the observer gain can be statically determined from the known $\hat{a}(t)$.

7.4 ADAPTIVE OBSERVERS FOR MULTI-OUTPUT SYSTEMS

Although extension of the preceding single-input single-output theory to multi-input systems is fairly straightforward, it is rather more difficult to develop an adaptive observer design for multi-output systems. The major difficulty, common to all identification schemes, lies in an adequate parameterization of a linear multivariable system model and the prior information required to justify the choice of model. In the following, a procedure is outlined which may be viewed as a multi-input multi-output generalization of the minimal-realization one of Section 7.2.

The identification problem involves the selection of a state–space system model $\{C, A, B\}$ on the basis of given input–output data. A special class of models is described by $\{C, A + KC, B\}$ where C and A are known, (C, A) is an observable pair and K and B are parameters to be identified. Assuming the system can be described by such a model, we have

$$\begin{aligned} \dot{x} &= (A + KC)x + Bu \\ y &= Cx \end{aligned} \tag{7.39}$$

where $x \in R^n$, $u \in R^r$ and $y \in R^m$ or, equivalently

$$\dot{x} = (A + K_0C)x + (K - K_0)y + Bu \tag{7.40}$$

where $A + K_0C$ is a stable matrix. A full-order observer for the system (7.40) is

$$\begin{aligned} \dot{\hat{x}} &= (A + K_0C)\hat{x} + (\hat{K}(t) - K_0)y + B\hat{u} + w \\ \hat{y} &= C\hat{x}. \end{aligned} \tag{7.41}$$

By (7.40) and (7.41) one obtains the error equations

$$\begin{aligned} \dot{e} &= (A + K_0C)e + (\hat{K}(t) - K_0)y + (\hat{B}(t) - B)u + w \\ &= (A + K_0C)e + \Phi(t)y(t) + \Psi(t)u(t) + w(t) \\ \hat{y}(t) - y(t) &= Ce(t). \end{aligned} \tag{7.42}$$

It is seen that (7.42) is similar in form to the single-output Equation (7.8). Defining an $n \times n\ (r+m)$ matrix E as

$$E = [E_1, E_2, \ldots, E_{r+m}]$$

where each E_i is an $(n \times n)$ matrix satisfying

$$\begin{aligned} \dot{E}_i &= (A + K_0 C)E_i + Iy_i, \qquad i = 1, 2, \ldots, m \\ \dot{E}_{m+j} &= (A + K_0 C)E_{m+j} + Iu_j, \quad j = 1, 2, \ldots, r \end{aligned} \tag{7.43}$$

the vector $w(t)$ and the adaptive laws for updating $\hat{K}(t)$ and $\hat{B}(t)$ are derived as

$$\begin{aligned} w(t) &= -E(t)E'(t)C'[\hat{y}(t) - y(t)] \\ \dot{L}(t) &= E'(t)C'[\hat{y}(t) - y(t)] \end{aligned} \tag{7.44}$$

where L is an $(nr + nm)$ vector obtained by arranging the columns of $\hat{K}(t)$ and $\hat{B}(t)$ one under the other. Substituting (7.44), the observer Equation (7.42) becomes

$$\dot{\hat{x}} = (A + K_0 C)\hat{x} + (\hat{K}(t) - K_0)y + \hat{B}(t)u - EE'C(\hat{y} - y). \tag{7.45}$$

Hence, it is necessary to generate an $(n \times n)$ matrix $E_i(t)$ corresponding to each input and output to synthesize this adaptive observer for multivariable systems.

7.5 ADAPTIVE OBSERVERS WITH EXPONENTIAL RATE OF CONVERGENCE

An alternative adaptive state reconstruction scheme is now considered where, by using an equivalent but structurally different representation of observer, the state reconstruction process is separated from the adaptation process. This simplifies the convergence analysis and leads to a globally exponential (rather than asymptotic) stable adaptive observer.

We consider the completely controllable and observable single-input single-output system

$$\begin{aligned} \dot{x} &= Ax + bu, \quad x(0) = x_0 \\ y &= c'x. \end{aligned} \tag{7.46}$$

Let D be the stable coefficient matrix of the completely observable pair (D, c') associated with the full-order observer

$$\dot{z} = Dz + gy + hu, \quad z(0) = z_0 \tag{7.47}$$

which has the solution

$$z(t) = \int_0^t \exp\{D(t-\tau)\}[Iy(\tau), Iu(\tau)]\, d\tau \begin{bmatrix} g \\ h \end{bmatrix} + \exp(Dt)z_0. \qquad (7.48)$$

By (7.46) and (7.47) we have exact state reconstruction, i.e. $z(t) \equiv x(t)$ if g, h and z_0 satisfy

$$D + gc' = A, \quad h = b, \quad z_0 = x_0. \qquad (7.49)$$

Therefore, using the notation $M(t)$ for the convolution integral in (7.48) and $\tilde{D} = \mathrm{diag}\,\{0, 0, D\}$, we obtain from (7.48) the alternative representation of the system state

$$x(t) = [M(t), I]p(t) \qquad (7.50)$$

$$\dot{p} = \tilde{D}p, \quad p_0 = \begin{bmatrix} g \\ h \\ x_0 \end{bmatrix}. \qquad (7.51)$$

Here the parameter vector $p(t)$ contains the information of the system parameters through (7.49), while the information of the system signals is contained in $M(t)$. The fact that $x(t)$ is a *linear* function of the system parameters in (7.50) is what makes this representation particularly suitable for the construction of a stable adaptive observer.

If D is chosen, for example in the phase variable form (2.56) and c' is likewise specified, then g is uniquely defined from the eigenvalues of A and (7.49). In turn this specifies A itself and hence the basis in which the system is to be considered. With c' and A specified, b and h are uniquely determined from (7.46), (7.49) and the numerator polynomial of the system transfer function.

Corresponding to the state representation (7.50), the adaptive observer (7.48) is rewritten more concisely as

$$\hat{x}(t) = [M(t), I]\hat{p}(t) \qquad (7.52)$$

$$\dot{M} = DM + [Iy(t), Iu(t)], \quad M_0 = 0. \qquad (7.53)$$

Here $\hat{p}(t)$ is the parameter vector to be adapted such that if $\Delta p \triangleq \hat{p} - p$ and $e \triangleq \hat{x} - x$

$$\lim_{t\to\infty} \Delta p(t) = 0, \quad \lim_{t\to\infty} e(t) = 0. \qquad (7.54)$$

Also, it follows from (7.50) and (7.52) that

$$e(t) = [M(t), I]\, \Delta p(t). \qquad (7.55)$$

In order that satisfactory adaptation may be achieved, the system is required

to be sufficiently excited by the input $u(t)$ in the sense of the following proposition.

Proposition 7.1 *The input $u(t)$ is said to be sufficiently exciting if there exist $t_1 > 0$, $\rho > 0$ such that*

$$R(t) \geqslant \rho I \quad \text{for } t \geqslant t_1. \tag{7.56}$$

It can be shown [K35] that a periodic $u(t)$ containing at least n distinct frequencies is always sufficiently exciting.

The adaptive observer is designed on the basis of the adaption of the parameter vector $\hat{p}(t)$ in the following law.

Adaptive law: The parameter vector $\hat{p}(t)$ is adapted according to

$$\dot{\hat{p}} = \tilde{D}\hat{p} - \kappa(t)\{R(t)\hat{p} + r(t)\}, \quad \hat{p}_0 \text{ arbitrary} \tag{7.57}$$

where $R(t)$ and $r(t)$ are defined by

$$\dot{R} = -qR - \tilde{D}'R - R\tilde{D} + [M(t), I]'cc'[M(t), I], \quad R_0 = 0 \tag{7.58}$$

$$\dot{r} = -qr - \tilde{D}'r - [M(t), I]'cy(t), \qquad r_0 = 0 \tag{7.59}$$

with $q > 2|\text{Re}\ \lambda_i(D)|$.

The scalar adaptive gain $\kappa(t)$ is given by

$$\kappa(t) = \gamma + |\dot{\mu}(t)| \tag{7.60}$$

where

$$\dot{\mu} = -\lambda\mu + \sqrt{n/2}\cdot(|y(t)| + |u(t)|), \quad \mu_0 = 0. \tag{7.61}$$

λ is assumed to satisfy $D + D' \leqslant -2\lambda I_n < 0$ and $\gamma > 0$ is an arbitrary constant.

Bringing together Proposition 7.1 and the Adaptive law (7.57) to (7.61) we are now in a position to establish the main result of this section, namely the global exponential convergence and stability of the adaptive observer (7.52).

Theorem 7.2 *Let $u(t)$ be any continuous input function which is sufficiently exciting in accordance with Proposition 7.1. Further, let the adaptive gain $\kappa(t)$ be given by the adaptive law (7.57) to (7.61). Then the adaptive observer, defined by (7.52), (7.53) and (7.56) to (7.61), has the following properties.*

(i) *The adaptive law (7.57) is globally asymptotically stable in the sense of Lyapunov with respect to $\hat{p}_0$, the stability being exponential with rate not less than $\gamma\rho$.*
(ii) *The state reconstruction error $e(t)$ converges to zero exponentially with rate not less than $\gamma\rho$.*

Proof (i) Since $u(t)$ is continuous it is obvious by (7.46), (7.60) and (7.61) that the observer has a unique continuous solution. By (7.51) and (7.57), $R(t)p(t) + r(t) \equiv 0$ and we have

$$\Delta\dot{p} = [\tilde{D} - \kappa(t)R(t)]\,\Delta p, \quad \Delta p_0 = \hat{p}_0 - p_0. \tag{7.62}$$

Evaluating the time derivative of the Lyapunov function $V(t) = \Delta p'(t)\,\Delta p(t)$ along the motion of (7.62) and using $\tilde{D}' + \tilde{D} \leqslant 0$ yields

$$\dot{V}(t) \leqslant 2\kappa(t)\,\Delta p'(t)R(t)\,\Delta p(t) \leqslant 0. \tag{7.63}$$

For $t \geqslant t_1$, it follows from (7.56) and (7.60) that

$$\dot{V}(t) \leqslant -2\gamma p V(t). \tag{7.64}$$

Therefore (7.62) and hence (7.57) is globally exponentially stable in the sense of Lyapunov with respect to $\hat{p}_0$ with rate not less than γp.

(ii) From (7.55) we have

$$e'(t)e(t) \leqslant V(t)[1 + \|M(t)\|]^2. \tag{7.65}$$

By application of the Schwarz inequality to (7.53) and the use of (7.61) it may be shown [K35] that $\|M(t)\| \leqslant \mu(t)$ so that (7.65) implies that

$$e'(t)e(t) \leqslant V(t)[1 + \mu(t)]^2. \tag{7.66}$$

Thus, defining $\tilde{V} = V[1 + \rho\mu(t)]^2$ and using the inequality $(1 + \mu)\cdot\rho \leqslant (1 + \rho\mu)(1 + \rho)$, one obtains

$$e'(t)e(t) \leqslant \tilde{V}[(1 + \rho)/\rho]^2. \tag{7.67}$$

Evaluation of $\dot{\tilde{V}}/\tilde{V}$ and use of (7.64), $t \geqslant t_1$, gives

$$\dot{\tilde{V}}(t) \leqslant -2\rho\left\{\kappa(t) - \frac{\dot{\mu}(t)}{1 + \rho\mu(t)}\right\}\tilde{V}(t), \quad t \geqslant t_1. \tag{7.68}$$

Equations (7.68) and (7.60) imply $\dot{\tilde{V}}/\tilde{V} \leqslant -2\gamma\rho$, such that

$$e'(t)e(t) \leqslant \tilde{V}(t_1)[(1 + \rho)/\rho]^2 \exp\{-2\gamma\rho(t - t_1)\}, \quad t \geqslant t_1. \tag{7.69}$$

Because $u(t)$ is continuous, $\mu(t_1)$ and consequently $\tilde{V}(t_1) = V(t_1)\cdot[1 + \rho\mu(t_1)]^2$ are bounded, such that $e(t) \to 0$ exponentially with rate $\gamma\rho$. Q.E.D.

In contrast to the adaptive observers of earlier sections, the convergence properties of the adaptive observer (7.52) hold for arbitrary system dynamics (including unstable systems) as well as arbitrary (not necessarily uniformly bounded) system inputs. These properties are essentially due to the fact that $\kappa(t)$ is adjusted according to signal level. Thus, by (7.55), convergence of $e(t)$ is obtained in the presence of unbounded $M(t)$ (signals) since the adaptation $\Delta p(t) \to 0$ is made sufficiently rapid compared with increasing $\|M(t)\|$.

7.6 LINEAR FEEDBACK CONTROL USING AN ADAPTIVE OBSERVER

It is now proposed to incorporate the adaptive observer of Section 7.5 in a closed-loop feedback control system. The adaptive observer (7.52) is particularly appropriate for two reasons. In the first place, it allows the algebraic separation of linear state feedback control and state reconstruction which has proven fundamental in the various feedback control schemes of earlier chapters. Secondly, the fact that stability of the observer does not require the system input and output signals to be bounded greatly facilitates the proof of the stability (signal boundedness) of the overall closed-loop system.

We consider a single-input linear state feedback control law

$$u(t) = f'x(t) + u_c(t) \tag{7.70}$$

where $u_c(t)$ is a continuous command input. Application of (7.70) to (7.46) results in the closed-loop system

$$\dot{x}^* = (A + bf')x^* + bu_c \tag{7.71}$$

if the system state $x^*(\cdot)$ were available for feedback control. It is assumed that sufficient knowledge of the system parameters in terms of A and b is available such that $A + bf'$ is stabilizable for some choice of feedback gain matrix f'.

In the event that $x(\cdot)$ is only partially available in the measurements (7.46), its replacement by the state estimate $\hat{x}(\cdot)$, generated by the adaptive observer (7.52), results in the useful separation of linear state feedback and state reconstruction.

Theorem 7.3 *Let the state feedback control law (7.70) give rise to a stable closed-loop system (7.71). Further, consider the feedback control law*

$$u(t) = f'\hat{x}(t) + u_c(t) \tag{7.72}$$

where $\hat{x}(t)$ is generated by the adaptive observer of Section 7.5. Then, if the command input $u_c(t)$ is continuous and guarantees $u(t)$ to be sufficiently exciting in the sense of Proposition 7.1, (7.72) gives rise to a closed-loop system, which is also globally asymptotically stable in the sense of Lyapunov with respect to the initial estimate $\hat{p}_0$ of the adaptive observer.

Proof (i) The overall closed-loop system, involving (7.46), (7.52), (7.53), (7.61) and (7.57) to (7.59), comprises a set of non-linear ordinary differential equations in x, M and μ. Existence of a unique solution for *all* $t, \sigma \leqslant t \leqslant \infty$, can be proven by contradiction [K35] by showing that continuity and satisfaction of any local Lipschitz condition extends to $0 \leqslant t \leqslant \infty$.

(ii) By part (i) above, $u(t)$ is continuous and by the assumption that $u_c(t)$ is

sufficiently exciting, Theorem 7.2 applies. Let $\Delta x \triangleq x - x^*$, then, substitution of (7.72) in (7.46), and (7.71) yield

$$\Delta \dot{x} = (A + bf')\,\Delta x + bf'e(t), \quad \Delta x_0 = 0. \tag{7.73}$$

If the initial conditions are $\hat{p}_0 = p_0$, then $e(t) \equiv 0$ and we have $\Delta x(t) \equiv 0$ as the equilibrium solution of (7.73). For the case $\hat{p}_0 \neq p_0$, Theorem 7.2 assures that $e(t)$ is bounded and vanishes exponentially independent of signal level. Since $A + bf'$ is a stability matrix, this implies uniform boundedness of $\Delta x(t)$ as well as $\lim_{t\to\infty} \Delta x(t) = 0$. By further detailed continuity arguments Lyapunov stability with respect to $\hat{p}_0$ follows. Q.E.D. □

It is stressed that the global Lyapunov stability of the resulting non-linear closed-loop control system makes no assumption on the stability of the original open-loop system, nor on the convergence rate and the dynamics of the non-linear adaptive observer. The only persistent assumption is that the external command $u_c(t)$ makes the system input $u(t)$ sufficiently exciting in the sense of Proposition 7.1.

7.6.1 Adaptive control by asymptotic feedback matrix synthesis

Throughout this section it has been tacitly assumed that adequate knowledge of the system matrices A and b is available for stabilizing feedback control design. Where the system parameter matrices A and b are completely unknown, one can however generate estimates of A and b from the adaptive observer parameter estimate

$$\hat{p}(t) \triangleq \begin{bmatrix} \hat{g}(t) \\ \hat{h}(t) \\ \exp(Dt)\hat{x}_0 \end{bmatrix} \tag{7.74}$$

through (7.49) and (7.51), as

$$\hat{A}(t) = D + \hat{g}(t)c', \quad \hat{b}(t) = \hat{h}(t). \tag{7.75}$$

The parameter estimation errors $\Delta A \triangleq \hat{A} - A$ and $\Delta b = \hat{b} - b$ are asymptotically stable (uniformly bounded) in the sense of Lyapunov by virtue of Theorem 7.2.

All of which brings us to the construction of the adaptive estimate $\hat{f}(t)$ of the feedback matrix f, based on the estimates $\hat{A}$ and $\hat{b}$ generated by the adaptive observer. For this synthesis problem the basic requirement is that $(\hat{A}, \hat{b}) \to (A, b)$ as $t \to \infty$ implies that $\hat{f} \to f$ as $t \to \infty$, where f is such that $A + bf'$ is a stability matrix.

Since the pair $(\hat{A}(t), \hat{b}(t))$ may be instantaneously uncontrollable or

unstabilizable, at time $t = t_1$, say, the synthesis of $\hat{f}(\cdot)$ based on $(\hat{A}(\cdot), \hat{b}(\cdot))$ will not exist at time $t = t_1$.

In order to avoid such singularities, we consider the following asymptotic synthesis of the adaptive feedback matrix

$$\hat{f} = \hat{f}(\hat{A}, \hat{b}, \hat{q}) \tag{7.76}$$

$$\dot{\hat{q}} = l(\hat{A}, \hat{b}, \hat{q}), \quad \hat{q}_0 \text{ arbitrary.} \tag{7.77}$$

Here Equation (7.76) represents the algebraic part of the feedback matrix synthesis. Equation (7.77) represents the asymptotic part, introduced to remove singularities, so that in the steady state

$$0 = l(A, b, q) \tag{7.78}$$

results in the desired asymptotically achieved equality

$$f = \hat{f}(A, b, q(A, b)) \tag{7.79}$$

where $A + bf'$ is a stability matrix.

An example of the asymptotic adaptive synthesis (7.76) and (7.77) is the Lyapunov stability technique of selecting the feedback control matrix f as

$$f = -Pb \tag{7.80}$$

where P satisfies the algebraic Lyapunov equation

$$A'P + PA = -I_n. \tag{7.81}$$

Replacing (A, b) in (7.80) and (7.81) by $(\hat{A}, \hat{b})$ and introducing suitable dynamics for generating $\hat{P}$ instead of solving the corresponding algebraic equation, results in the asymptotic synthesis of the adaptive feedback matrix

$$\hat{f}(\hat{A}, \hat{b}, \hat{P}) = -\hat{P}\hat{b} \tag{7.82}$$

$$\dot{\hat{P}} = \hat{A}'\hat{P} + \hat{P}\hat{A} + I_n, \quad \hat{P}_0 \text{ arbitrary.} \tag{7.83}$$

Equations (7.82) and (7.83) are readily observed to be a special case of the general synthesis (7.76) and (7.77).

Proceeding with the general synthesis problem, we have instead of the feedback control law (7.72) the linear adaptive feedback control law

$$u(t) = \hat{f}'(\cdot)\hat{x}(t) + u_c(t) \tag{7.84}$$

where $\hat{f}$ and $\hat{x}$ are generated by the adaptive observer of Section 7.5. Similar to Theorem 7.3, the algebraic separation of adaptive state feedback control and adaptive state reconstruction is secured in the following theorem.

Theorem 7.4 *Let the unknown system (7.46) be controlled by the adaptive feedback control law (7.84) where $\hat{f}$ and $\hat{x}$ are generated by the adaptive*

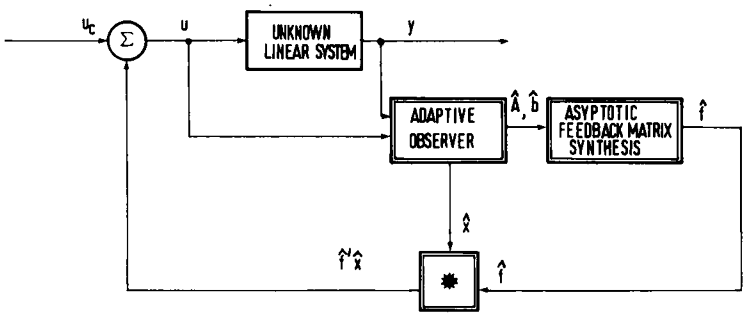

Fig. 7.3 Structure of the adaptive control system.

observer of Section 7.5. Then, if the command input $u_c(t)$ is continuous, uniformly bounded and sufficiently exciting in the sense of Proposition 7.1, (7.84) gives rise to a closed-loop system which is globally asymptotically stable in the sense of Lyapunov with respect to $\hat{p}_0$ and $\hat{q}_0$.

Proof The proof which in many respects follows closely that of Theorem 7.3 is to be found in [K36].

The structure of the complete adaptive feedback control system is depicted in Fig. 7.3.

7.7 NOTES AND REFERENCES

Interest in the possibility of adaptive identification and control of unknown linear systems was stimulated by the publication of a model reference approach, based on Lyapunov stability, by Butchart and Shackloth [B23] and Parks [P3]. Good surveys of this approach to stable adaptive observer and controller design are given by Narendra *et al.* [N2], [N3], [N4].

The original contribution to an adaptive observer theory using the model reference approach is due to Carroll and Lindorff [C1] while the treatment of Section 7.2 follows Narendra *et al.* [K39], [L6], [N3]. In Section 7.3 the simplified observer design for systems in non-minimal canonical form follows Lüders and Narendra [L7] (see also Ichikawa [I4]). The extension of the adaptive observer design to multi-output systems in Section 7.4 is due to Morse [M19]. An alternative transfer-function approach is presented by Anderson [A6].

The implicit or parameterized adaptive observer of Section 7.5 was introduced by Kreisselmeier [K33], [K34] and is closely related to an earlier system identification scheme proposed by Lion [L5]. The present exposition is patterned on the development of Kreisselmeier [K35] where an earlier

assumption of signal boundedness is no longer required. This is particularly advantageous for the proof of global exponential stability of the observer-based adaptive controller of Section 7.6. Theorem 7.4 is due to [K36] and it can be shown [K37] that the adaptive observer and asymptotic feedback control matrix synthesis is also locally exponentially stable for systems (a) without external command inputs, (b) of higher order than originally assumed, (c) containing slight non-linearities and time-variation effects and (d) subject to unknown system and measurement disturbances. Extension of the theory to multi-output systems is, however, unclear at present.

Other contributions include an exponentially stable implicit adaptive observer of minimal order by Nuyan and Carroll [N11] and explicit observers, based on hyperstability theory, by Hang [H3] and Suzuki and Andoh [S21] for continuous and discrete-time systems respectively. An adaptive observer for polynomial-matrix system models is presented by Elliott and Wolovich [E1].

Chapter 8

Observer-based Compensation in the Frequency Domain

8.1 INTRODUCTION

Since the publication of [R10], much effort has been directed over the last decade to the investigation of what classical frequency response methods, suitably generalized, have to offer in the analysis and design of linear multivariable systems. Section 4.5 illustrated, in an introductory way, that the classical frequency–domain concepts of transfer function, return difference, etc., can be generalized and usefully related to the newer state–space techniques, previously discussed in some detail. Within the framework of complex-variable theory, a deeper examination is now undertaken of the complementary role multivariable frequency response and state–space methods have to play in observer-based compensation.

In Section 8.2, the return-difference matrix introduced in Section 4.5 is used to provide multivariable Nyquist-type characterizations of the relative stability and optimality of the minimal-order state observer.

Definitions and properties of poles and zeros of linear multivariable state–space systems are presented in Section 8.3. Roughly speaking, poles are associated with the internal dynamics of the system while zeros can be thought of as those frequencies at which transmission through the system to the external environment is blocked. The importance of system zeros in determining closed-loop system performance under high-gain output feedback is discussed in Section 8.4 and extended to observer-based controller design in Section 8.5.

Section 8.6 and Section 8.7 highlight some of the difficulties that may arise in the exclusive pursuit of time–domain methods of design for observer-based controllers from the point of view of system robustness (adequate stability margins in the face of model parameter variations) and controller instabilities.

8.2 OBSERVER RETURN-DIFFERENCE MATRIX PROPERTIES

Our first brief encounter with the system *return-difference matrix* description was for linear state feedback control systems in Section 4.5. It was noted that the return-difference matrix exhibits a fundamental relationship between the open-loop and closed-loop behaviour of multi-loop control systems. Viewed from a frequency–domain standpoint, the feedback nature of the deterministic minimal-order state observer of Chapter 2 and the stochastic observer-estimator of Chapter 6 may also be further illuminated by pursuing a dual matrix generalization of the Bode return-difference concept.

8.2.1 The minimal-order state observer

Consider, without loss of generality, the completely observable linear time-invariant state–space system described by

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(t) \tag{8.1}$$

$$y(t) = [I_m \quad 0]x(t). \tag{8.2}$$

The system (8.1) and (8.2) will usually also contain a control input vector, but this has been omitted for notational convenience.

By Definition 2.1, the state $x(t) \in R^n$ of the system may be asymptotically reconstructed by a minimal-order time-invariant state observer of the form

$$\dot{z}(t) = (A_{22} - V_2A_{12})z(t) + (A_{21} - V_2A_{11} + A_{22}V_2 - V_2A_{12}V_2)y(t) \tag{8.3}$$

$$\hat{x}(t) = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix} z(t) + \begin{bmatrix} I_m \\ V_2 \end{bmatrix} y(t). \tag{8.4}$$

In particular, by (8.4),

$$\hat{x}_2(t) = z(t) + V_2y(t). \tag{8.5}$$

Differentiation of (8.5) and the use of (8.3) yields the following alternative expression for the dynamical behaviour of the reconstructed state

$$\hat{x}_2(t) \in R^{n-m}$$

$$\dot{\hat{x}}_2(t) = A_{22}\hat{x}_2(t) + A_{21}y(t) + V_2(\dot{y}(t) - A_{11}y(t) - A_{12}\hat{x}_2(t)). \tag{8.6}$$

With reference to the original system description (8.1) and (8.2), the equivalent form of the observer (8.6) illustrates its role as a dynamic model for the unknown portion of the state $x_2(t)$. Errors in the modelling process are compensated by the comparison term $(\dot{y} - A_{11}y - A_{12}\hat{x}_2)$ weighted by an appropriate gain matrix V_2. It is noted, however, that unlike (8.3) and (8.4), this

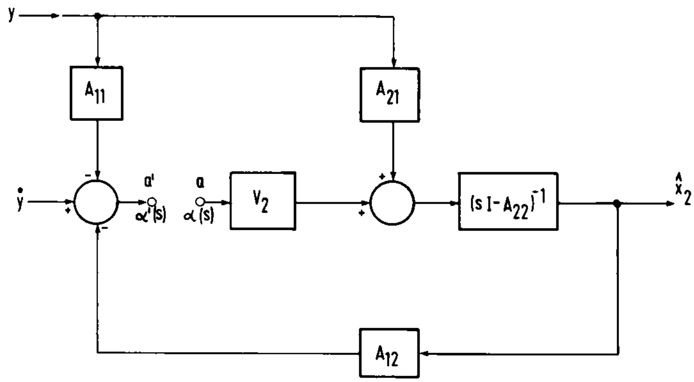

Fig. 8.1 Definition of observer return-difference.

form of observer is unsuitable *vis-à-vis* implementation due to the need to differentiate the measurements $y(t)$.

To further examine the feedback properties of the minimal-order observer consider the observer (8.6) exhibited in the frequency domain as in Fig. 8.1. Suppose that all the feedback loops are broken, as shown in Fig. 8.1, and a (deterministic) signal transform vector $\alpha(s)$ is injected at point a. The transform of the signal vector returned at a′ is then

$$-A_{12}(sI_{n-m} - A_{22})^{-1}V_2\alpha(s)$$

and the difference between the injected and returned signal-transform vectors is thus

$$[I_m + A_{12}(sI_{n-m} - A_{22})^{-1}V_2]\alpha(s) = F_2(s)\alpha(s) \tag{8.7}$$

where $F_2(s)$ is defined as the *return-difference matrix* of the minimal-order state observer. It is noted that the observer return-difference matrix $F_2(s)$ is the dual of the controller return-difference matrix defined in Section 4.5.1; namely,

$$F_1(s) = I_r - F(sI_n - A)^{-1}B \tag{8.8}$$

where F is an $(r \times n)$ controller gain matrix. Moreover, a little reflection on the previous analysis would confirm that the return-difference matrix of a full-order (Kalman) observer is given by

$$F_2(s) = I_m + C(sI_n - A)^{-1}G \tag{8.9}$$

where G is an $(n \times m)$ observer gain matrix.

It is of interest to characterize the (relative) stability of the minimal-order observer in terms of the return difference $F_2(s)$ of Equation (8.7). Using arguments dual to those of Section 4.5.1, one arrives at the fundamental

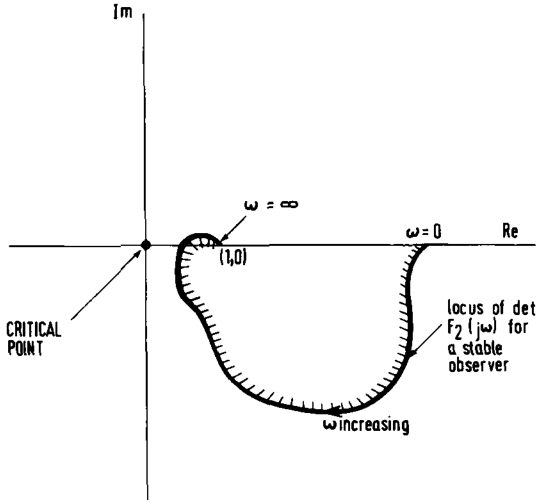

Fig. 8.2 Multivariable Nyquist-type stability criterion.

relation (cf. Equation (4.83))

$$\det F_2(s) = \frac{\text{closed-loop characteristic polynomial}}{\text{open-loop characteristic polynomial}} \tag{8.10}$$

If the system is open-loop stable, the open-loop characteristic polynomial will have no zeros in the closed right-half complex plane. Consequently, the closed-loop observer is stable if, and only if, $\det F_2(s)$ does not vanish in the right-half plane. Further, let D be a semi-circle centred on the origin, of large enough radius α in the right-half plane so as to enclose every right-half plane zero and pole of the determinants of the open-loop and closed-loop transfer functions.

Suppose D maps into a closed curve Γ in the complex plane under the mapping $\det F_2(s)$. Then the observer is closed-loop stable if no point within D maps on to the origin of the complex plane under the mapping $\det F_2(s)$. Thus, the observer is closed-loop stable if Γ does *not* enclose the origin of the complex plane. If $|\det F_2(s)| \to 1$ as $|s| \to \infty$, then, taking α as arbitrarily large, we can conveniently refer to Γ as the locus $\det F_2(j\omega)$. This gives the multi-loop Nyquist-type criterion shown in Fig. 8.2.

The minimum distance from the origin to the $\det F_2(j\omega)$ locus, like the classical gain and phase margin [P19], is a measure of the relative stability of the observer. Since

$$\det F_2(s) = \prod_{i=1}^{m} \rho_i(s) \tag{8.11}$$

where $\rho_i(s)$, $i = 1, 2, \ldots, m$, are the eigenvalues of $F_2(s)$, the Nyquist-type criterion can be restated as follows.

Theorem 8.1 *The minimal-order state observer* (8.6) *is closed-loop stable if none of the eigenvalue loci* $\rho_i(j\omega)$, $i = 1, 2, \ldots, m$, *of the return-difference matrix* $F_2(s)$, *defined by* (8.7) *and* (8.11), *enclose the origin of the complex plane.*

In other words, the loci of the eigenvalues $\rho_i(j\omega)$ of the return-difference matrix $F_2(j\omega)$, associated with a stable observer, will be somewhat similar in nature to that of det $F_2(j\omega)$ represented in Fig. 8.2.

8.2.2 The stochastic linear least-squares observer-estimator

The return-difference properties of the corresponding stochastic linear least-squares stationary estimation problem solution of Chapter 6 are now examined. It is recalled in Corollary 6.1 that the inaccessible part of the random state vector of the stationary linear system

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(t) + \xi(t) \tag{8.12a}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} I_{m_1} & 0 \\ C_{21} & C_{22} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \eta_2(t) \end{bmatrix} \tag{8.12b}$$

can be estimated in a minimum mean-square error sense by an $(n - m_1)$th order observer-estimator described by

$$\begin{aligned}\dot{\hat{x}}_2 = {} & A_{22}\hat{x}_2(t) + A_{21}y_1(t) + K_1[\dot{y}_1(t) - A_{11}y_1(t) - A_{12}\hat{x}_2(t)] \\ & + K_2[y_2(t) - C_{21}y_1(t) - C_{22}\hat{x}_2(t)]\end{aligned} \tag{8.13}$$

The estimator gain matrix $K \triangleq [K_1, K_2]$ is given by

$$K = (\Sigma_{22}\bar{C}' + \bar{S})\hat{R}^{-1} \tag{8.14}$$

where Σ_{22} satisfies the algebraic matrix Riccati equation

$$\begin{aligned}0 = {} & A_{22}\Sigma_{22} + \Sigma_{22}A'_{22} + Q_{22} \\ & - (\Sigma_{22}\bar{C}' + \bar{S})\hat{R}(\Sigma_{22}\bar{C}' + \bar{S})'\end{aligned} \tag{8.15}$$

in which

$$\bar{C} \triangleq \begin{bmatrix} A_{12} \\ C_{22} \end{bmatrix}, \quad \bar{S} \triangleq [Q_{21}, S_2], \quad \hat{R} \triangleq \begin{bmatrix} Q_{11} & S_1 \\ S'_1 & R_{22} \end{bmatrix} \tag{8.16}$$

represent appropriate partitions of the system parameter and noise covariance matrices.

In order to facilitate the transformation of the algebraic matrix Riccati Equation (8.15) into the frequency domain the following lemma is required.

Lemma 8.1 *If there exist symmetric matrices $Q \geqslant 0$ and $R > 0$ and a matrix S such that Σ is the unique symmetric positive definite solution of the matrix Riccati equation*

$$-A\Sigma - \Sigma A' + KRK' = Q \tag{8.17}$$

where

$$K = (\Sigma C' + S)R^{-1} \tag{8.18}$$

then the return-difference matrix

$$F(s) \triangleq I + C(sI - A)^{-1}K \tag{8.19}$$

satisfies the operator equation

$$F(s)RF'(-s) = R + G(s)S + S'G'(-s) + G(s)QG'(-s) \tag{8.20}$$

where

$$G(s) \triangleq C(sI - A)^{-1} \tag{8.21}$$

Proof Rewriting (8.17) as

$$(sI - A)\Sigma + \Sigma(-sI - A') + KRK' = Q$$

one has, upon premultiplication by $C(sI - A)^{-1}$ and postmultiplication by $(-sI - A')^{-1}C'$ that

$$\begin{aligned} &C\Sigma(-sI - A')^{-1}C' + C(sI - A)^{-1}\Sigma C' + C(sI - A)^{-1}KRK'(-sI - A')^{-1}C' \\ &\quad = C(sI - A)^{-1}Q(-sI - A')^{-1}C' \\ &\quad = G(s)QG'(-s) \end{aligned} \tag{8.22}$$

Eliminate Σ from (8.22) using (8.18) or

$$\Sigma C' = KR - S, \quad C\Sigma = RK' - S' \tag{8.23}$$

to give

$$\begin{aligned} &\{RK'(-sI - A')^{-1}C' - S'(-sI - A')^{-1}C'\} + \{\text{transpose}\} \\ &\qquad + C(sI - A)^{-1}KRK'(-sI - A')^{-1}C' = G(s)QG'(-s). \end{aligned} \tag{8.24}$$

Addition of R to both sides of (8.24), together with a little rearrangement, yields the desired result (8.20). Q.E.D. □

Bearing in mind that the estimator (8.13) is an $(n - m_1)$th-order system which estimates the subsystem state $x_2(\cdot) \in R^{n-m_1}$, let the $m \times (n - m_1)$

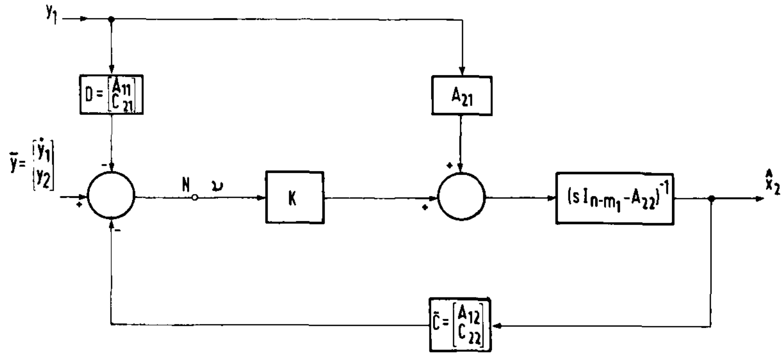

Fig. 8.3 Definition of estimator return-difference.

subsystem transfer–function matrix be

$$G(s) \triangleq \bar{C}(sI_{n-m_1} - A_{22})^{-1}. \tag{8.25}$$

Also, let the estimator return-difference matrix, determined by the gain matrix $K = [K_1, K_2]$, be

$$F(s) = I_m + \bar{C}(sI_{n-m_1} - A_{22})^{-1}K. \tag{8.26}$$

Then, by invoking Lemma 8.1, we may transform Equation (8.15) into the frequency domain to obtain

$$G(s)Q_{22}G'(-s) + \hat{R} + G(s)\bar{S} + \bar{S}'G'(-s) = F(s)\hat{R}F'(-s). \tag{8.27}$$

The definition of the minimal-order estimator return-difference matrix $F(s)$ in (8.26) is associated with viewing the estimator as a *feedback mechanism* in Fig. 8.3 where all the feedback loops are broken at the point N.

When all the feedback loops are closed, the input to the estimator is the *innovations process* $\nu(s)$ which is related to the "observations" $\bar{y}(s) - Dy_1(s)$ by

$$\nu(s) = \begin{bmatrix} sIy_1(s) \\ y_2(s) \end{bmatrix} - \begin{bmatrix} A_{11} \\ C_{21} \end{bmatrix} y_1(s) - \bar{C}(sI - A_{22})^{-1}[K\nu(s) + A_{21}y_1(s)] \tag{8.28}$$

or, using (8.25), (8.26) and (8.28),

$$\nu(s) = F^{-1}(s)\left\{\begin{bmatrix} sIy_1(s) \\ y_2(s) \end{bmatrix} - \begin{bmatrix} A_{11} \\ C_{21} \end{bmatrix} y_1(s) - G(s)A_{21}y_1(s)\right\}. \tag{8.29}$$

Thus $F^{-1}(s)$ is seen to act as a *whitening filter* on the derived "observations" $\bar{y}(s) - (D + GA_{21})y_1(s)$ in which the noise-free measurements y_1 have already been partially whitened by differentiation.

Also, from the system model (8.12), (8.16) and (8.25) we have that

$$\begin{bmatrix} sIy_1(s) \\ y_2(s) \end{bmatrix} - \begin{bmatrix} A_{11} \\ C_{21} \end{bmatrix} y_1(s) - GA_{21}y_1(s) = G(s)\xi_2(s) + \begin{bmatrix} \xi_1(s) \\ \eta_2(s) \end{bmatrix} \tag{8.30}$$

so that the left-hand side of (8.27) is the $m \times m$ para-Hermitian density matrix of the "observations" $\bar{y}(s) - (D + GA_{21})y_1(s)$ and may be denoted by $\Phi(s)$. Factorization of the spectral matrix $\Phi(s)$ in (8.27) as

$$F(s)\hat{R}F'(-s) = \Delta(s)\,\Delta'(-s) \tag{8.31}$$

where $\Delta(s)$ is either the spectral factor corresponding to stable $G(s)$ [M4] or the generalized spectral factor corresponding to unstable $G(s)$ [S7], results in

$$F(s) = \Delta(s)\hat{R}^{-1/2}. \tag{8.32}$$

The manner of derivation and final form (8.32) of the return-difference matrix $F(s)$ of the minimal-order estimator in terms of the factor of the power spectral density of the "observations" $\bar{y}(s) - (D + GA_{21})y_1(s)$ resembles that of the Kalman–Bucy filter (MacFarlane [M4], Shaked [S7]). There are, however, important differences.

The return-difference matrix $F(s)$, as specified by (8.32), depends on the power spectral density of $\dot{y}_1$ as well as y_2 and so requires knowledge of model parameter matrices. Also required is the covariance of the system disturbance ξ_1 as well as that of measurement noise η_2. For these reasons it is preferable that the return-difference matrix $F(s)$ be determined by the estimator gain matrix rather than vice versa as in the non-singular case [S7].

It is of interest to characterize the optimality of the estimator (8.13) in terms of the locus of $\det F(j\omega)$. Let $\mu(s)$ be a left-hand eigenvector of $F(s)$ corresponding to the eigenvalue $\rho(s)$, so that

$$\mu(s)F(s) = \mu(s)\rho(s). \tag{8.33}$$

Premultiply Equation (8.27) by $\mu(s)$ and postmultiply by $\mu'(-s)$ to obtain

$$\mu(s)G(s)Q_{22}G'(-s)\mu'(-s) + \mu(s)\{\hat{R} + G(s)\bar{S} + \bar{S}'G'(-s)\}\mu(-s) = \rho(s)\rho(-s)\mu(s)\hat{R}\mu'(-s) \tag{8.34}$$

Since the term in Q_{22} is positive semi-definite at $s = j\omega$, a necessary condition for optimality is

$$|\rho(j\omega)|^2 \geqslant \frac{\mu(j\omega)\{\hat{R} + G(j\omega)\bar{S} + \bar{S}'G'(-j\omega)\}\mu'(-j\omega)}{\mu(j\omega)\hat{R}\mu'(-j\omega)}. \tag{8.35}$$

Using

$$\det F(j\omega) = \prod_{i=1}^{m} \rho_i(j\omega) \tag{8.36}$$

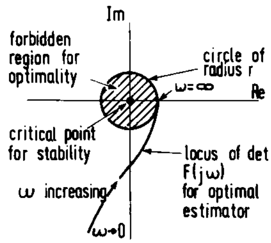

Fig. 8.4 Necessary condition for optimality satisfied by return-difference determinant.

the necessary condition for optimality (8.35) becomes

$$|\det F(j\omega)| \geqslant r \tag{8.37}$$

where $r > 0$ is the radius of a circle.

The graphical interpretation of (8.37), as displayed in Fig. 8.4, is that the complex-plane plot of $F(j\omega)$ must not penetrate the interior of the circle of radius r. Thus optimality of the estimator is seen to be a more stringent requirement than stability. It is left as an exercise to verify that in the special case in which all the measurements are noisy ($m_1 = 0$, $m_2 = m$), and the measurement and process noises are uncorrelated ($S = 0$), the forbidden region for optimality is the unit circle (see also MacFarlane [M4]).

8.3 POLES AND ZEROS OF LINEAR MULTIVARIABLE SYSTEMS

Consider the linear time-invariant state–space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{8.38}$$

where $x(t) \in R^n$, $u(t) \in R^r$ and $y(t) \in R^m$.

Taking one-sided Laplace transforms yields the set of equations

$$sx(s) - x(0) = Ax(s) + Bu(s)$$
$$y(s) = Cx(s) \tag{8.39}$$

Eliminating the initial state $x(0)$ by setting it to zero, one has, as in Section 4.5, that the input and output transform vectors are related by

$$y(s) = G(s)u(s) \tag{8.40}$$

where

$$G(s) \triangleq C(sI - A)^{-1}B \tag{8.41}$$

is the *proper transfer-function matrix* relating input and output transforms.

Alternatively, retaining the initial state vector $x(0)$, the complex frequency set of Equations (8.39) may be rewritten as

$$\begin{bmatrix} (sI - A) & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ y(s) \end{bmatrix} \tag{8.42}$$

The matrix

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \tag{8.43}$$

is known as the *system matrix* (Rosenbrock [R10]) and contains all the information about the system which is needed to describe its behaviour including that of the system zeros. Just as we think of the transfer-function matrix $G(s)$ as an *external* system description, the system matrix $P(s)$ which exhibits the internal structure associated with the state-space model (8.38) can be thought of as an *internal* description.

Clearly, $P(s)$ conveys more information about the system than $G(s)$ which, as we have observed in Section 1.4, represents only the completely controllable and completely observable part of the system (8.38). In general, a larger set of zeros is defined via $P(s)$ than via $G(s)$. The zeros associated with the transfer-function matrix $G(s)$ are often called the *transmission zeros* in that they block the transmission of an input $u(t) = u_0 e^{st}$, $t \geqslant 0$ to the output $y(t)$; that is, $y(t) \equiv 0$, $t > 0$. Correspondingly, the zeros associated with the system matrix $P(s)$ are known as the system *invariant zeros*. If, as is assumed in the sequel, the system (8.38) is completely controllable and completely observable, the set of invariant zeros reduces to the set of transmission zeros. We mention, in passing, however, that in general the set of invariant zeros of $P(s)$ contains, in addition to the set of transmission zeros of $G(s)$, the sets of *output decoupling zeros* and *input decoupling zeros*. These latter sets of zeros are respectively associated with the unobservable and uncontrollable system modes displayed in the canonical structure theorem of Theorem 1.11. That is, in accordance with Theorem 1.12, they correspond to the loss of rank of the matrices

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} \quad \text{and} \quad [sI - A, -B]$$

respectively.

Although our discussion of system zeros will for the most part centre on *square* systems, we digress for a moment to consider *non-square* systems;

systems of which the number of inputs is *not* equal to the number of outputs. If, for instance, $m > r$ and $G(s)$ in (8.41) is of full rank r, different sets of zeros can be computed for the various square $r \times r$ subsystems formed from the system by selecting different sets of r outputs. If follows from the fact (Kouvaritakis and MacFarlane [K32]) that the zeros of the non-square system are the intersection of the sets of zeros for all these $r \times r$ subsystems that a general *non-square* transfer–function matrix $G(s)$ almost always has *no* transmission zeros.

For the rest of this section, we confine our attention to the set of invariant zeros (transmission zeros) associated with the square ($r = m$) system matrix description (8.43) of the completely controllable and completely observable system (8.38). Fortunately, for the case $m > r$, any non-square system can be converted to a square system by the procedures known as "squaring up" of the inputs or "squaring down" of the outputs. The case $r > m$ is even more favourable in that one need only select any m-dimensional subset of controls. In either case $m > r$ or case $m < r$, the only conveat is that the squaring or selection procedure must not introduce unstable (right-half plane) invariant zeros for reasons that will become apparent in Section 8.4.

Regarding the square transfer–function matrix $G(s)$ in (8.41)

$$\det\{G(s)\} = \det\{C(sI - A)^{-1}B\} \tag{8.44}$$

which, using Schur's formula for partitioned determinants (Appendix A), becomes

$$\det\{G(s)\} = \frac{\det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}}{\det(sI - A)} = \frac{z(s)}{p_0(s)}. \tag{8.45}$$

This leads to (long awaited) definitions of the poles and invariant zeros of a square system (8.38) via $G(s)$ and $P(s)$.

Definition 8.1 The poles of the square system (8.38) (open-loop transfer–function matrix (8.41)) are the zeros of the characteristic polynomial

$$p_0(s) \triangleq \det(sI - A). \tag{8.46}$$

Definition 8.2 The invariant zeros of the square system (8.38) are the zeros of the *invariant zero polynomial*

$$z(s) \triangleq \det\{P(s)\} = \det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}. \tag{8.47}$$

Definition 8.1 is a formal statement to the effect, observed in Section 1.4, that the poles of $G(s)$ are simply the eigenvalues of the minimal system matrix A. The adjective *invariant* describing the system zeros of Definition 8.2 refers to

the fact that such zeros are invariant with respect to a range of non-singular transformations and feedback control laws.

Theorem 8.2 *The system zeros, as defined in Definition 8.2, are invariant under the following set of transformations:*

(*i*) *non-singular co-ordinate transformations in the state-space;*
(*ii*) *non-singular transformations of the inputs;*
(*iii*) *non-singular transformations of the outputs;*
(*iv*) *linear state feedback to the inputs;*
(*v*) *linear output feedback to the inputs.*

Proof The proof is based on elementary operations on the determinant of $P(s)$ and is discussed in Kouvaritakis and MacFarlane [K31].

To conclude our discussion of system invariant zeros on a geometric note, we introduce the notion of *invariant-zero directions.*

Definition 8.3 Associated with each invariant zero s_1 of the system (8.38), there exists a pair of vectors $x_i(0)$ and g_i such that the composite vector

$$\begin{bmatrix} x_i(0) \\ g_i \end{bmatrix}$$

defined as the invariant zero-direction vector, lies in the kernal or null space of $P(s_i)$.

That is,

$$\begin{bmatrix} s_i I - A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} x_i(0) \\ g_i \end{bmatrix} = 0. \tag{8.48}$$

It is remarked that

$$\begin{bmatrix} v_i \\ Fv_i \end{bmatrix}$$

where v_i is the eigenvector corresponding to the ith eigenvalue of $(A + BF)$, in Theorem 4.1 is an invariant-zero direction vector if it lies in the kernal of $[C, 0]$. Physically, the input vector and state vector time variations, corresponding to an identically zero system output, may be regarded as associated with rectilinear motions along the zero directions in the input and state spaces respectively. In particular, the rectilinear motion in the state–space occurs in the null space of C.

8.4 CLOSED-LOOP POLE ASSIGNMENT UNDER HIGH-GAIN OUTPUT FEEDBACK

The invariant zeros, interpreted geometrically in terms of the system matrices A, B and C in the preceding section, are now shown to play a central role in determining closed-loop system behaviour under high-gain feedback.

Consider the application of the linear output feedback control law

$$u(t) = -Fy(t), \quad F \triangleq kI_m \tag{8.49}$$

to the square system (8.38). For arbitrarily large values of the scalar gain k, such a control law is termed *high-gain feedback*. The accompanying closed-loop system matrix takes the form

$$A_{CL} = A - kBC. \tag{8.50}$$

As $k \to \infty$, the poles of the closed-loop system (8.50) vitally depend on the geometric structure of BC as to which poles remain finite and which poles tend to infinity.

This structure we now examine where, without loss of generality, it may be assumed that B and C are each of full rank m. It is also assumed, for the present, that rank $BC = m$. The matrix BC will accordingly bear a spectral decomposition of the form

$$\begin{aligned} BC &= U\Lambda_{BC}V \\ &= [W \quad M]\begin{bmatrix} \Lambda & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix}\begin{bmatrix} Z \\ N \end{bmatrix} \end{aligned} \tag{8.51}$$

where Λ is a diagonal matrix of non-zero eigenvalues λ_i, $i = 1, \ldots, m$, of BC, and U and V have as their columns and rows respectively the appropriate eigenvectors and reciprocal eigenvectors of BC. Since M and N contain those eigenvectors and reciprocal eigenvectors of BC which correspond to the set of zero eigenvalues we have, by definition of the eigenvalue problem (Appendix A), that

$$(BC)M = 0_{n,n-m} \tag{8.52}$$

$$N(BC) = 0_{n-m,n} \tag{8.53}$$

If (8.52) is premultiplied by a left-inverse of B and (8.53) is postmultiplied by a right-inverse of C, one has

$$CM = 0_{m,n-m} \tag{8.54}$$

$$NB = 0_{n-m,m} \tag{8.55}$$

Moreover, $UV = I_m$ and (8.51) imply that

$$MN = I_{n-m}. \tag{8.56}$$

In the light of this structure for BC it is appropriate to exhibit the closed-loop system matrix (8.50) in the basis:

$$\begin{aligned}\bar{A}_{CL} &= V(A - kBC)U \\ &= VAU - k\Lambda_{BC} \\ &= \begin{bmatrix} Z \\ N \end{bmatrix} A[W \quad M] - k\begin{bmatrix} \Lambda & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix} \\ &= \begin{bmatrix} (ZAW - k\Lambda) & ZAM \\ NAW & NAM \end{bmatrix}. \end{aligned} \tag{8.57}$$

The closed-loop system poles (characteristic frequencies) are the zeros of the closed-loop characteristic polynomial

$$\det(sI_n - \bar{A}_{CL}) = \det\begin{bmatrix} (sI_m - ZAW + k\Lambda) & -ZAM \\ -NAW & (sI_{n-m} - NAM) \end{bmatrix}. \tag{8.58}$$

Applying Schur's formula (Appendix) to this partitioned determinant it follows that the closed-loop poles are determined by

$$\det(sI_m - ZAW + k\Lambda)\det(-NAW(sI_m - ZAW + k\Lambda)^{-1}ZAM + sI_{n-m} - NAM) = 0 \tag{8.59}$$

where $\det(sI_{n-m} - ZAW + k\Lambda)$ is assumed to be non-zero and

$$\det(sI_{n-m} - NAM)\det(-ZAM(sI_{n-m} - NAM)^{-1}NAW + sI_m - ZAW + k\Lambda) = 0 \tag{8.60}$$

where $\det(sI_{n-m} - NAM)$ is assumed to be non-zero. Equations (8.59) and (8.60) are, of course, valid for all k.

If we let k become artibrarily large, Equations (8.59) and (8.60) reduce to

$$\det(-NAW(sI_m + k\Lambda)^{-1}ZAM + sI_{n-m} - NAM) = 0 \tag{8.61}$$

and

$$\det(sI_m + k\Lambda) = 0. \tag{8.62}$$

Moreover for finite s and infinite values of k, these equations are replaced by

$$\det(sI_{n-m} - NAM) = 0 \tag{8.63}$$

and

$$s = -k\lambda_i, \quad i = 1, \ldots, m. \tag{8.64}$$

It is observed that as $k \to \infty$, $n - m$ of the closed-loop poles tend to the finite

zeros of (8.63); hence the appellation *finite zeros*. The remaining set of m closed-loop poles tend to infinity as $k \to \infty$ in accordance with (8.64). Thus the solutions (8.64) are often described as *infinite zeros* in that they may be regarded as attracting the m poles that tend to infinity as $k \to \infty$. This separation of the gain dependent closed-loop poles into those which are attracted with increasing gain to the system finite zeros and those which tend along asymptotes (8.64) to the "infinite zeros" is the basis of a direct multivariable generalization of the graphical root-locus method (Kouvaritakis and Edmunds [K30]).

Recall that an underlying assumption of the above analysis is that BC is of full rank m. It can be shown (Kouvaritakis and MacFarlane [K31]) by a simple extension of this analysis that if the rank of BC is $m - d$ ($d \geqslant 0$), we have the intuitively satisfying result that as $k \to \infty$, $m + d$ closed-loop poles tend to the infinite zeros

$$s = -k\lambda_i, \quad i = 1, \ldots, m + d \tag{8.65}$$

and the remaining $n - m - d$ closed-loop poles tend to the finite zeros of*

$$\det(sNM - NAM) = 0. \tag{8.66}$$

Geometrically, the kernel of C in (8.54) and the left null space of B in (8.55), together with the eigenstructure of A, uniquely define the invariant zeros associated with the system matrix $P(s)$ in (8.47). In fact, if a left transformation L and a right transformation R, defined respectively as

$$L = \begin{bmatrix} N & 0 \\ B\dagger & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} M & C\dagger & 0 \\ 0 & 0 & I \end{bmatrix}$$

are applied to the system matrix $P(s)$ in (8.43), it is readily demonstrated that the finite zeros of (8.66) are *precisely* the invariant zeros of $P(s)$..

Since $n - m - d$ closed-loop poles are attracted to these zeros with increasing gain, the importance of an earlier admonition in Section 8.3 that the system invariant (transmission) zeros should not lie in the right-half complex plane is now clear. Systems unfortunate enough to possess right-half plane zeros are somewhat innocuously described as *non-minimum phase* systems. The name derives from the fact that a scalar transfer function with a zero s_0 in the right-half plane has the same gain but greater phase shift than if the zero were at $-\bar{s}_0$ in the left-half plane. Such systems impose severe structural restrictions on the control action that can usefully be applied. Since, by Theorem 8.2,

* A polynomial matrix of the form $sE - A$ is known as a *pencil of matrices*. It is termed a *regular* matrix pencil if and only if it is square and $\det(sE - A) \neq 0$; otherwise it is called a *singular* matrix pencil.

system zeros are invariant with respect to feedback, these restrictions are in no way alleviated by feedback control. Bearing in mind the intimate structural relation between the system zeros and the coupling matrices B and C, the only means of removing unwanted zeros is to restructure these coupling matrices by selecting appropriate different sets of inputs and/or outputs.

We conclude our study of finite zeros and infinite zeros by remarking that for the latter zeros, the asymptotic directions of the root loci that vanish at infinity depend on the system *Markov parameters* CA^iB, $i = 0, 1, 2, \ldots$ (Kouvaritakis and Edmunds [K30]).

8.5 OBSERVERS FOR HIGH-GAIN FEEDBACK SYSTEMS WITH INACCESSIBLE STATE

Instead of using static output feedback, it is timely to alternatively consider the deployment of a state observer for high-gain feedback systems with inaccessible state. As has been witnessed in successive chapters, the appeal of an observer is that the overall closed-loop pole assignment problem separates into the two simpler independent problems of pole assignment of the system as if the state were accessible and assignment of the observer poles. In so far as the closed-loop poles of the original system tend with increasing gain to the finite and infinite zeros, it is natural to inquire if such a separation principle extends to include finite and infinite zeros. A formal confirmation of our intuitive feeling that it does is now embarked on for both full-order and minimal-order state observers.

But consider first the application of the linear state feedback control law

$$u(t) = Fx(t) \tag{8.67}$$

to the system (8.38). For our present purpose, Equation (8.67) may be viewed a state measurement vector y_F which is directly fed back to the inputs. Thus, by (8.66), the system finite zeros are the solutions of

$$\det(sNM - NAM) = 0 \tag{8.68}$$

where, by (8.54) and (8.55), M and N satisfy

$$FM = 0, \quad NB = 0. \tag{8.69}$$

Also, in cognizance of the last remark of Section 8.4, the infinite zeros of the system depend on the Markov parameters FA^iB, $i = 0, 1, 2, \ldots$. It will be helpful to characterize the system under feedback control (8.67) in an open-loop sense as $S(A, B, F)$, just as we might refer to the system (8.38) as $S(A, B, C)$.

8.5.1 The full-order state observer

Owing to the incomplete availability of the system state the feedback control policy (8.67) is unrealizable. Consider, instead, the linear control law

$$u(t) = F\hat{x}(t) \tag{8.70}$$

where the state estimate $\hat{x}(t) \in R^n$ is generated by the full-order state observer

$$\dot{\hat{x}}(t) = (A - GC)\hat{x}(t) + Gy(t) + Bu(t). \tag{8.71}$$

The overall system $S(\hat{A}, \hat{B}, \hat{C})$ is described by the composite set of equations

$$\dot{x}_a = \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \hat{A}x_a + \hat{B}u \tag{8.72}$$

$$y_F = \hat{C}x_a \tag{8.73}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ GC & A - GC \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ B \end{bmatrix}, \quad \hat{C} = [0, F]. \tag{8.74}$$

In order to determine the finite zeros of the augmented system $S(\hat{A}, \hat{B}, \hat{C})$, we require, similar to (8.69), two full rank matrices $\hat{M}$ and $\hat{N}$ to satisfy the conditions

$$\hat{C}\hat{M} = 0, \quad \hat{N}\hat{B} = 0. \tag{8.75}$$

By inspection, two suitable candidates are

$$\hat{M} = \begin{bmatrix} M & I_n \\ M & 0_n \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} 0 & N \\ I_n & -I_n \end{bmatrix}. \tag{8.76}$$

Thus, analogous to (8.68), the finite zeros of the augmented system $S(\hat{A}, \hat{B}, \hat{C})$ are the solutions of

$$\det\,(s\hat{N}\hat{M} - \hat{N}\hat{A}\hat{M}) = \det\left(s\begin{bmatrix} NM & 0 \\ 0 & I_n \end{bmatrix} - \begin{bmatrix} NAM & NGC \\ 0 & A - GC \end{bmatrix}\right) = 0. \tag{8.77}$$

That is, the finite zeros of $S(\hat{A}, \hat{B}, \hat{C})$ are the zeros of

$$\det\,(sNM - NAM) = 0 \tag{8.78}$$

and

$$\det\,(sI - A + GC) = 0. \tag{8.79}$$

This is an interesting result. It is observed that the finite zeros of the composite system consist of the set of finite zeros provided, as expected, by the original system $S(A, B, F)$, and a set of zeros given by the observed poles.

Turning to the determination of infinite zeros for the augmented system $S(\hat{A}, \hat{B}, \hat{C})$, it is recalled that they depend on the Markov parameters $\hat{C}\hat{A}^i\hat{B}$. Now, by (8.74), we have that the condition

$$\hat{A}^i\hat{B} = \begin{bmatrix} A^iB \\ A^iB \end{bmatrix} \tag{8.80}$$

holds true for $i = 0$ and $i = 1$, but also if it holds for $i = k$ then it will hold for $i = k + 1$ too; hence this condition holds true for all i. Using (8.74), it is readily deduced that

$$\hat{C}\hat{A}^i\hat{B} = FA^iB. \tag{8.81}$$

In other words, the infinite zeros of the augmented system $S(\hat{A}, \hat{B}, \hat{C})$ are identical to those of the original system $S(A, B, F)$; they correspond to the asymptotic behaviour of the original system and are in no way affected by the observer.

8.5.2 The minimal-order state observer

Consider the linear system

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = [I_m, \quad 0]x = x_1 \tag{8.82}$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$ and A and B are conformally partitioned. If the state of (8.82) were completely available, a feedback control law of the form

$$y_F = [F_1, F_2]x \tag{8.83}$$

might usefully be applied. In this case, full rank matrices M and N which satisfy (8.69) take the special form

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} -F_1^{-1}F_2 \\ I_{n-r} \end{bmatrix} \tag{8.84}$$

$$N = [N_1, N_2] = [-B_2B_1^{-1}, I_{n-r}] \tag{8.85}$$

where it is assumed that F and B are of full rank so that by a suitable renumbering of the states it is always possible to obtain non-singular F_1 and B_1. Thus the finite zeros of $S(A, B, F)$, by (8.68), (8.84) and (8.85) are the solutions of

$$\det(sNM - NAM) = \det(s(N_1M_1 + I_{n-r}) + N_1A_{11}M_1 + N_1A_{12} + A_{21}M_1 + A_{22}) = 0. \tag{8.86}$$

By Definition 2.1, the actual inaccessible part of the state, $x_2(t)$, of the system (8.82) may be asymptotically reconstructed by a minimal-order observer of the

form

$$\dot{z} = (A_{22} - V_2 A_{12})z + (A_{21} - V_2 A_{11} + A_{22} V_2 - V_2 A_{12} V_2)y + (B_2 - V_2 B_1)u \tag{8.87}$$

$$\hat{x}_2 = z + V_2 y. \tag{8.88}$$

The linear feedback law takes the corresponding form

$$y_F = [F_1 \quad F_2]\begin{bmatrix} x_1 \\ \hat{x}_2 \end{bmatrix} \tag{8.89}$$

Proceeding along similar lines to those of the full-order observer case, the system-observer combination is described by the augmented system $S(\hat{A}, \hat{B}, \hat{C})$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = \left[\begin{array}{c|c|c} A_{11} & A_{12} & 0 \\ \hline A_{21} & A_{22} & 0 \\ \hline A_{21} - V_2 A_{11} + A_{22} V_2 - V_2 A_{12} V_2 & 0 & A_{22} - V_2 A_{12} \end{array}\right] \times \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_2 - V_2 B_1 \end{bmatrix} u \tag{8.90}$$

$$y_F = [F_1 + F_2 V_2 \quad 0 \quad F_2]\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}. \tag{8.91}$$

By inspection it is easy to verify that suitable candidates to satisfy (8.75) are

$$\hat{M} = \begin{bmatrix} -F_1^{-1} F_2 & 0 \\ I_{n-r} & -I_{n-r} \\ I_{n-r} + V_2 F_1^{-1} F_2 & 0 \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} N_1 & N_2 & 0 \\ V_2 & -I_{n-r} & I_{n-r} \end{bmatrix}. \tag{8.92}$$

Thus the finite zeros of the augmented system $S(\hat{A}, \hat{B}, \hat{C})$ are determined as the roots of the equation

$$\det(s\hat{N}\hat{M} - \hat{N}\hat{A}\hat{M})$$

$$= \det\left(s\left[\begin{array}{c|c} N_1 M_1 + I_{n-r} & -I_{n-r} \\ \hline 0 & I_{n-r} \end{array}\right] - \left[\begin{array}{c|c} N_1 A_{11} M_1 + A_{21} M_1 + N_1 A_{12} + A_{22} & -N_1 A_{12} + A_{22} \\ \hline 0 & A_{22} - V_2 A_{12} \end{array}\right]\right) = 0. \tag{8.93}$$

which by (8.86) is equivalent to

$$\det(sNM - NAM) = 0 \quad \text{and} \quad \det(sI - A_{22} + V_2 A_{12}) = 0. \tag{8.94}$$

Once again, the finite zeros of the (open-loop) system-observer combination comprises the set of finite zeros of the original system and the set of poles of the observer.

The infinite zeros of the system-observer combination $S(\hat{A}, \hat{B}, \hat{C})$ in (8.90) and (8.91) are determined by the Markov parameters $\hat{C}\hat{A}^i\hat{B}$. Again, using induction, it is easy to show that

$$\begin{aligned}\hat{C}\hat{A}^i\hat{B} &= [F \quad F_2]\left[\begin{array}{c|c} A^i & 0 \\ \hline 0 & (A_{22} - V_2 A_{12})^i \end{array}\right]\begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= FA^iB. \end{aligned} \tag{8.95}$$

Just as in the full-order observer case, the infinite zeros of the composite system-observer are precisely those of the original system with feedback gain matrix F, and are unaffected by the minimal-order observer.

8.6 ROBUST OBSERVER-BASED CONTROLLER DESIGN

The term *robustness* has gained wide currency in the control studies literature. It seems almost destined to supplant optimality as arbiter of "goodness" of control systems design. As with the latter quality, the appeal of robustness is as much due to its generality as that attributable to any single virtue. Broadly speaking, it refers to the preservation of desirable system properties such as stability, insensitivity to parameter variations and noise attenuation in the face of undesirable environmental effects such as induced parameter variations, noise, etc. Our present use of the word robustness is restricted to that of relative stability; if a closed-loop system remains stable under large parameter variations, it is deemed robust.

It was observed in Section 4.6 that the accessible state optimal linear regulator possesses impressive gain and phase margins. Where the system state is inaccessible and a state reconstructor (observer) is inserted in the feedback loop, it is natural to inquire if such favourable robustness properties are preserved and this we now do.

We consider the square linear minimum-phase system (8.38) under direct state feedback control as in Fig. 8.5 and under observer-based feedback control as in Fig. 8.6. The controller transfer function matrices $H_1(s)$ and $H_2(s)$ include the linear regulator as a special case ($H_1(s) = I_m$, $H_2(s) = F$) and further allow the inclusion of dynamic elements such as integrators and time-delay elements. It is recalled from Section 4.5 that the closed-loop transfer

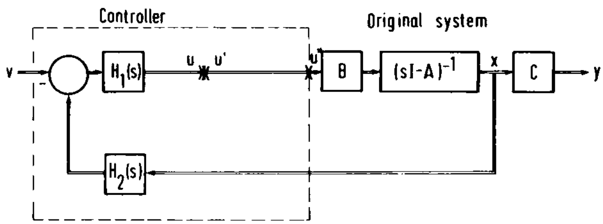

Fig. 8.5 State feedback controller.

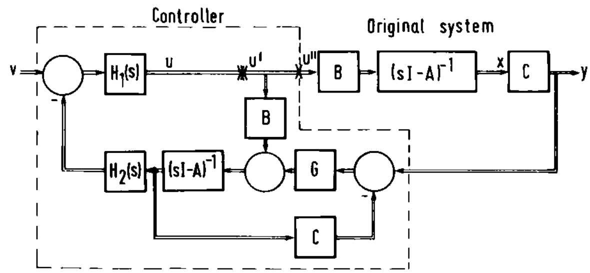

Fig. 8.6 Observer-based controller.

function matrices from the external input v to the state x are identical in both configurations due to the fact that the observer error dynamics are uncontrollable from, and therefore unexcited by, the input u'.

8.6.1 Comparison of robustness via the return-difference

Given the open-loop characteristic polynomial of the system (8.38), a second method of characterizing the closed-loop system stability in either configuration, suggested by Equation (4.83) or (8.10), is by way of the return-difference matrix. Section 4.5 reminds us that the main part of the determination of the return-difference matrix at point XX is the calculation of the loop transfer-function matrix from signal u' to signal u (loops broken at point XX); otherwise known as the system *return-ratio matrix* (MacFarlane [M3]). Following Section 4.5, it is left as a simple exercise to verify that the return-ratio matrix, defined at point XX, is identical in both implementations. Again, the reason is that the observer error dynamics are not excited by the inputs u' to the system with loop broken at point XX in Fig. 8.6.

Thus far, identical closed-loop transfer function matrices and return-difference matrices are reported for the state feedback and observer-based control configurations, promising equal robustness. The fly in the ointment is in the following lemma.

Lemma 8.2 *The return-ratio matrices from the control signal u'' to the control signal u' (loops broken at point X) are, in general,* different *in the two configurations. They are identical if the observer dynamics satisfy*

$$G[I + C(sI - A)^{-1}G]^{-1} = B[C(sI - A)^{-1}B]^{-1}. \tag{8.96}$$

Proof With reference to Fig. 8.5, the return-ratio matrix from $u''(s)$ to $u'(s)$ of the direct state feedback configuration is given by

$$u' = -H_1H_2\Phi Bu'' \tag{8.97}$$

where $\Phi(s) \triangleq (sI - A)^{-1}$ is known as the system *resolvent matrix.*

The corresponding relationship for the observer-based implementation of Fig. 8.6 is

$$\hat{x} = (\Phi^{-1} + GC)^{-1}\{Bu' + GC\Phi Bu''\}. \tag{8.98}$$

Thus, since $u' = -H_1H_2\hat{x}$, one obtains

$$u' = -H_1H_2(\Phi^{-1} + GC)^{-1}\{Bu' + GC\Phi Bu''\}. \tag{8.99}$$

Application of the matrix inversion lemma (Appendix) to the $(\phi^{-1} + \mathrm{GC})^{-1}$ term in (8.99) yields

$$\begin{aligned} u' &= -H_1H_2[\Phi - \Phi G(I + C\Phi G)^{-1}C\Phi]\{Bu' + GC\Phi Bu''\} \\ &= -H_1H_2\Phi[B - G(I + C\Phi G)^{-1}C\Phi B]u' \\ &\quad -H_1H_2\Phi G(I + C\Phi G)^{-1}C\Phi Bu''. \end{aligned} \tag{8.100}$$

If (8.96) is satisfied, the first term on the right-hand side of (8.100) vanishes and the second term reduces to (8.97). That is, if (8.96) is satisfied the return-ratio matrices at point X are identical in both configurations. Q.E.D. □

Using similar methods, it is readily shown (Doyle and Stein [D10]) that the return-ratio matrices are identical at an arbitrary breaking point if a modified version of condition (8.96) is satisfied.

By Lemma 8.2, the difference in return-ratio matrices at point X means that, in general, the robustness of the state feedback scheme does not extend to the observer-based implementation unless (8.96) holds. This is because, unlike before, the observer error dynamics do get excited in response to inputs u'' with loops broken at X.

More importantly, these difficulties are avoided if one chooses an observer gain $G(q)$ such that as the scalar gain parameter $q \to \infty$

$$\frac{G(q)}{q} \to BW \tag{8.101}$$

for any non-singular matrix W. Then Equation (8.96) will be satisfied asymptotically as $q \to \infty$. The limiting poles of the full-order observer are the

roots of the characteristic polynomial $\Psi(s)$, described by (8.9), (8.10) and (8.101) as

$$\Psi(s) = \det(sI - A)\det[I + qC(sI - A)^{-1}BW]. \qquad (8.102)$$

As $q \to \infty$ (cf. Section 8.4) P of these roots will tend towards the zeros of

$$\Psi(s) = \det(sI - A)\det[C(sI - A)^{-1}B]$$

$$= \det\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \qquad (8.103)$$

that is, by Definition 8.2, the system invariant zeros. Hence, the imperative that the system be minimum phase or contain no zeros in the right-half complex plane. The remaining $n - P$ observer poles tend in a dual sense to Section 8.4. to the infinite zeros of the system. It is thus clear that the commonly suggested approach of Chapter 4 of making *all* the roots of the observer error dynamics artibrarily fast is generally the wrong thing to do from the point of view of securing adequate closed-loop system robustness.

It is emphasized that the choice of observer gain (8.101) to satisfy (8.96) asymptotically will only work for systems, discussed in Section 8.4, which are stabilizable by high-gain output feedback. By contrast, it can be shown (Doyle and Stein [D10]) that the following Kalman-type filter gain works for all controllable, observable, minimum-phase systems:

$$G(q) = \Sigma(q)C'R^{-1} \qquad (8.104)$$

where $\Sigma(q)$ satisfies the algebraic Riccati equation

$$A\Sigma + \Sigma A' + Q(q) - \Sigma C'R^{-1}C\Sigma = 0 \qquad (8.105)$$

for which

$$Q(q) = Q_0 + q^2BVB' \qquad (8.106)$$

$$R = R_0 \qquad (8.107)$$

and Q_0 and R_0 are noise covariance matrices for the nominal system.

Finally, we note that our methods apply equally well to control schemes based on the minimal-order state observer. In the latter case, overall robustness properties may be deduced in terms of the gain matrix V_2 associated with the observer coefficient matrix $A_{22} - V_2A_{12}$.

8.7 UNSTABLE OBSERVER-BASED CONTROLLERS AND HOMEOPATHIC INSTABILITY

Consider again the closed-loop observer-based control scheme of Fig. 8.6

where $H_1(s) = I$, $H_2(s) = F$ and $v(t) = 0$. In accordance with Theorem 1.15 and Theorem 1.16 our consistent design philosophy has been to choose the controller gain F and observer gain G such that $A + BF$ and $A - GC$ are stability matrices and, more importantly, the overall closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{8.108}$$

is asymptotically stable.

But what of the stability of the observer-based controller itself? The stability of the controller is determined by the transfer–function matrix from $y(s)$ to $u''(s)$. That is, dropping superscripts on $u(s)$, we have either from Fig. 8.6 or the controller equations

$$\dot{\hat{x}}(t) = (A - GC + BF)\hat{x}(t) + Gy(t) \tag{8.109}$$

$$u(t) = F\hat{x}(t) \tag{8.110}$$

that the matrix transfer function representation of the controller is

$$u(s) = \mathfrak{C}_c(s)y(s) \tag{8.111}$$

where

$$\mathfrak{C}(s) = F[sI - (A - GC + BF)]^{-1}G \tag{8.112}$$

$$= F\left\{\frac{\operatorname{adj}[sI - (A - GC + BF)]}{\det[sI - (A - GC + BF)]}\right\}G \tag{8.113}$$

It follows from (8.113) that the controller $\mathfrak{C}(s)$ will be *unstable* if the composite matrix $(A - GC + BF)$ has any eigenvalues in the right-half complex plane.

Theorem 8.3 *There exists a class of linear systems (8.38) such that one or more eigenvalues of $(A - GC + BF)$ of the observer-based feedback controller may lie in the right-half complex plane though all the eigenvalues of $A + BF$ and all the eigenvalues of $A - GC$ are designed to lie in the left-half plane, and even though the system pairs (A, B), (A, C) are, respectively, completely controllable and completely observable.*

Theorem 8.4 *There exists a class of linear systems (8.38) exhibiting* homoeopathic instability *whereby the overall system (8.108) is asymptotically stable* only if *the observer-based feedback controller is itself unstable.*

The proofs of Theorem 8.3 and Theorem 8.4 are omitted but are given, via a general second-order example by Johnson [J3], [J4]. That a stable closed-loop system may consist of subsystems (including controller subsystems) which are themselves unstable is, of course, well known in control system

studies. The significance of Theorem 8.3 rests in the fact that a designer may unknowingly construct an unstable observer-based controller in order to stabilize the overall system. In this event, effective off-line frequency-response testing of the controller circuits is precluded in that auxiliary feedback loops are required to stabilize the unstable controller. Moreover, after installation, a momentary disconnection of the controller from the system will lead to unstable behaviour.

Frequently, a controller that is itself stable can be chosen from among the many different controllers which will stabilize the open-loop system (8.38), and so none of the aforecited difficulties arise. Theorem 8.4 is a particularly striking result in that for a special class of systems (and not a pathological one either) the open-loop system (8.38) can *only* be stabilized by an unstable observer-based controller. The term "homoeopathic instability", due to Johnson [J4], is borrowed from a school of medical thought (homoeopathy) which advocates consideration of curative agents which have characteristics resembling the malady being treated.

As we might expect, the findings of Theorem 8.3 and Theorem 8.4 extend *mutatis mutandis* to controllers based on the minimal-order state observer.

8.8 NOTES AND REFERENCES

The application of the return-difference matrix concept to the investigation of stability and optimality of minimal-order observers in Section 8.2 is the dual of that of feedback control discussed in Section 4.5. Our treatment of the return-difference matrix properties of the stochastic linear least-squares observer-estimator follows an extension of MacFarlane to the singular case by O.Reilly [O22]. An alternate frequency-domain approach to the singular estimation problem that is related to asymptotic optimal root loci is considered by Shaked and Bobrovsky [S8].

Of the several different (not all equivalent) definitions of poles and zeros of linear multivariable systems, the first comprehensive treatment is by Rosenbrock [R10]. The so-called Rosenbrock system matrix is introduced in that work and serves, in many ways, as a valuable natural link between internal state–space and external transfer–function representations of the system. Our state–space treatment of Section 8.3 is based on the work of MacFarlane and his collaborators, in particular MacFarlane and Karcanias [M7]; for further study of system poles and zeros and numerous other references see also the reprint volume [M6]. A brief discussion of poles and zeros from a strict transfer–function point of view is presented in Chapter 9.

The analysis of Section 8.4 in which system zeros are seen to play a key role in determining closed-loop system behaviour under high-gain feedback

follows that of Kouvaritakis and MacFarlane [K31]; additional discussion of finite and infinite zeros, high-gain feedback, inverse systems and asymptotic root loci is to be found in [M6] and Owens [O29]. It is mentioned in passing, however, that for a feedback system to have good gain margins, it is necessary that the set of finite zeros and the set of infinite zeros lie in the left half of the complex plane. The deployment of full-order and minimal-order state observers for high-gain systems in Section 8.5. is patterned on the recent exposition by Kouvaritakis [K29].

The fact that observer-based compensation schemes may have disappointing stability margins seems to have been first pointed out by Rosenbrock ([R10], p. 198). Attention to this issue of robustness in observer-based controller design is more recently drawn by Doyle [D9] and Doyle and Stein [D10]. The exposition of the problem, with measures for recovering full-state regulator robustness, in Section 8.6 follows that of [D10]. A broader up-to-date design prospective on robust multivariable feedback control is to be found in the special issue [S2]; see especially Doyle and Stein [D11] and Lehtomaki *et al.* [L1].

The possibility that a stabilizing feedback controller may itself be unstable is raised by Wonham ([W17], p. 65) and Johnson [J3]; the discovery that there is a special class of unstable systems that cannot be stabilized unless the feedback controller is itself unstable, is independently made by Youla *et al.* [Y4] and Johnson [J4]. Section 8.7 is based on the treatment of Johnson [J3], [J4]. The delineation of a general class of linear systems possessing this property of homoeopathic instability is an area for further research. Other system theoretic aspects of homoeopathic systems are at present unknown.

Chapter 9

Observer-based Compensation for Polynomial Matrix System Models

9.1 INTRODUCTION

In Chapter 4 and Chapter 8 we saw that the analysis and design of linear multivariable state feedback and observer-based controllers could be carried out using either time–domain state–space techniques or frequency response methods based on complex-variable theory. A third method allied to both, and consequently serving as a natural and unifying link between transfer–function methods and the newer state–space techniques, is that employing the *differential operator representation*, $\{N(D), Q(D), R(D)\}$, defined by the equations.*

$$N(D)z(t) = Q(D)u(t) \tag{9.1a}$$

$$y(t) = R(D)z(t) \tag{9.1b}$$

where $z(t) \in R^q$ is called the *partial state*, $u(t) \in R^r$ is the input and $y(t) \in R^m$ is the output. $N(D)$, $Q(D)$ and $R(D)$ are polynomial matrices of consistent dimensions in the differential operator $D = d/dt$, and $N(D)$ is non-singular to ensure a unique solution.

If we assume that the initial conditions of the partial state $z(t)$ and all its derivatives are zero, we immediately obtain upon taking Laplace transforms of (9.1) the system representation.

$$N(s)z(s) = Q(s)u(s) \tag{9.2a}$$

$$y(s) = R(s)z(s). \tag{9.2b}$$

Accordingly, the differential operator D and the Laplace operator s may be

* A more general system representation is $\{N(D), Q(D), R(D), W(D)\}$ where (9.1b) contains a transmission term, viz. $y(t) = (R(D)z(t) + W(D)u(t))$. Henceforth we assume $y(t)$ in (9.1b) to be replaced by $y(t) - W(D)u(t)$ if $W(D) \neq 0_{mr}$.

used interchangeably in our subsequent discussion of the *polynomial matrix model* (9.1) or (9.2).

It is noted that the polynomial system model (9.1) is a generalization of the state equation (8.39) or $\{DI - A, B, C\}$, defined by

$$(DI - A)x(t) = Bu(t)$$
$$y(t) = Cx(t). \tag{9.3}$$

Equivalently, the complete polynomial system model (9.2) may be put in the compact form

$$\begin{bmatrix} N(s) & -Q(s) \\ R(s) & 0 \end{bmatrix} \begin{bmatrix} z(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ y(s) \end{bmatrix} \tag{9.4}$$

whence the system matrix $P(s)$ of the state–space system (8.43) is observed to be a special case.

Viewed as an input–output or external system description, the transfer–function relation is deduced from (9.2) to be

$$y(s) = T(s)u(s) \tag{9.5}$$

where

$$T(s) = R(s)N^{-1}(s)Q(s) \tag{9.6}$$

It is recalled from Section 1.4 that when a transfer–function matrix is irreducible (possesses no cancellation of poles and zeros) it corresponds to a completely controllable and completely observable state–space system or minimal realization. In Section 9.2 these equivalences are further examined via the more general differential operator system representation (9.1). We also develop concomitant generalized state–space concepts of controllability and observability.

A polynomial matrix or transfer–function approach to linear feedback system regulation is pursued in Section 9.3 and Section 9.4 without any explicit use of the time–domain notion of state. Given a controllable polynomial system (9.1), it is shown that arbitrary closed-loop pole assignment may therefore be achieved. The system zeros, although cancellable by closed-loop system poles, otherwise remain unaltered by linear partial state feedback. Where the system partial state is inaccessible but the polynomial system model (9.1) is observable, a frequency–domain counterpart of the linear function observer of Chapter 3 is developed in Section 9.4. The transfer–function matrix of the overall closed-loop observer-based control system is quite the same as that of the closed-loop partial state feedback system, a result deduced in Section 4.5 and Section 8.6 for state–space systems.

9.2 SOME PROPERTIES OF THE POLYNOMIAL SYSTEM MODEL

The notion of system equivalence (similarity) discussed in Section 1.4 whereby different state–space realizations result in the same transfer–function matrix can be extended to the polynomial matrix system description (9.1).

Definition 9.1 Any two differential operator systems of the form (9.1),

$$\{N_1(D), Q_1(D), R_1(D)\} \quad \text{and} \quad \{N_2(D), Q_2(D), R_2(D)\}$$

are equivalent if

$$\{N_2(D), Q_2(D), R_2(D)\} = \{U_L(D)N_1(D)U_R(D), U_L(D)Q_1(D), R_1(D)U_R(D)\}$$

for any pair $\{U_L(D), U_R(D)\}$ of *unimodular matrices.**

Theorem 9.1 *Any two differential operator systems are equivalent if and only if they are both equivalent to the same (or equivalent) state–space representation(s).*

Proof See [W10], p. 141, for a constructive proof in the form of an algorithm.

In view of Theorem 9.1 it is evident that the various definitions and conditions of observability and controllability, made in Chapter 1 in connection with pure state–space systems, can be extended to more general differential operator representations.

Theorem 9.2 *Any equivalent state–space representation of the differential operator system (9.1) is completely controllable (completely observable) if and only if any one of the following equivalent conditions is satisfied:*

(*i*) *$N(D)$ and $Q(D)(R(D))$ are relatively left (right) prime;*

(*ii*) $\operatorname{rank}[N(D), Q(D)] = q\left(\operatorname{rank}\begin{bmatrix} N(D) \\ R(D) \end{bmatrix} = q\right) \quad \forall s \in \mathbb{C}$

(*iii*) *any* greatest common left (right) divisor† $G_L(D)$ $(G_R(D))$ *of* $N(D)$ *and* $Q(D)(R(D))$ *is unimodular;*

---

* A *unimodular matrix*, $U(D)$, is defined as any matrix which can be obtained from the identity matrix, I, by a finite number of elementary row and column operations on I. The determinant of a unimodular matrix is therefore a non-zero scalar and, conversely, any polynomial matrix whose determinant is a non-zero scalar is a unimodular matrix.

† A *greatest common right divisor* (gcrd) of two matrices $N(D)$ and $R(D)$ with the same number of columns is any matrix $G_R(D)$ with the properties:

(i) $G_R(D)$ is a right divisor of $N(D)$ and $R(D)$; i.e. there exist polynomial matrices $\bar{N}(D)$ and $\bar{R}(D)$ such that $N(D) = \bar{N}(D)G_R(D)$, $R(D) = \bar{R}(D)G_R(D)$;

(ii) if $G_{R1}(D)$ is any other right divisor of $N(D)$ and $R(D)$, then $G_{R1}(D)$ is a right divisor of $G_R(D)$. A greatest common left divisor (gcld) $G_L(D)$ can also be defined *mutatis mutandis.*

(*iv*) *there exist relatively right* (*left*) *prime polynomial matrices* $\bar{X}(D)$ *and* $\bar{Y}(D)(X(D)$ *and* $Y(D)$ *of appropriate dimensions such that the Bezout identity*

$$N(D)\bar{X}(D) + Q(D)\bar{Y}(D) = I_q \tag{9.7a}$$

$$(X(D)N(D) + Y(D)R(D) = I_q) \tag{9.7b}$$

is satisfied.

Proof See Rosenbrock [R10] p. 71 or Wolovich [W10] p. 153.

Conditions (i) and (ii) of Theorem 9.2 specialize to those of Theorem 1.12 for state–space systems. As a corollary to condition (iii), we have that all uncontrollable (unobservable) system modes are equal to the zeros of det $G_L(D)$ (det $G_R D$)). The second Bezout identity of condition (iv) will prove useful in the observer-based compensator design of Section 9.4. Henceforth, we shall say that the differential operator system is controllable (observable) if any equivalent state–space representation is completely controllable (completely observable).

To complete our brief review of some of the main connections between differential operator systems, state–space systems and transfer–function representations, we have the following theorem.

Theorem 9.3 *Any rational* $(m \times r)$ *transfer–function matrix* $T(s)$ *can be (non-uniquely) represented as*

$$T(s) = R(s)N_R^{-1}(s) + E(s) \tag{9.8}$$

where

(*i*) $N_R(s)$ *and* $R(s)$ *are relatively right prime, and*

(*ii*) $R(s)N_R(s)^{-1}$ is strictly proper, *i.e.* $R(s)N_R^{-1}(s) \to 0$ *as* $s \to \infty$, *and* $E(s)$ *is a polynomial matrix.*

Proof See Rosenbrock [R10] p. 101 or Wolovich [W10] p. 159.

Corollary 9.1 $T(s)$ *can alternatively be (non-uniquely) factored as*

$$T(s) = N_L^{-1}(s)U(s) + E(s) \tag{9.9}$$

where N_L, U *and* E *obey conditions analogous to* (*i*) *and* (*ii*) *of Theorem 9.3.*

In the sequel, it is assumed that the system is described by the strictly proper rational $(m \times r)$ transfer–function matrix $T(s)$

$$T(s) = R(s)N_R^{-1}(s) \tag{9.10}$$

where $R(s)$ and $N_R(s)$ are relatively right prime and $R(s)$'is of lower column degree‡ than $N_R(s)$. Fortunately, it is in general always possible to subtract the polynomial matrix $E(s)$ from $T(s)$ in (9.8) to obtain a strictly proper transfer–function matrix.

The matrix factorization (9.10) directly implies the differential operator realization

$$N_R(D)z(t) = u(t) \tag{9.11}$$

$$y(t) = R(D)z(t) \tag{9.12}$$

Thus the differential operator representation (9.11) and (9.12) enjoys two advantages over its state–space counterpart: not only does it imply and is implied by a transfer–function matrix factorization, but the partial state $z(t) \in R^q$ is of lower dimension than the corresponding state $x(t) \in R^n$. Moreover, since rank $[N_R(s), I_q] = q \forall s \in \mathbb{C}$, it is immediate from Theorem 9.2 that the differential operator representation (9.11) and (9.12) is controllable. Any differential operator realization (9.11) and (9.12) is *minimal* if it is also observable. Equivalently, in accordance with Theorem 9.2, the matrix-fraction transfer–function $T(s)$ in (9.10) is said to be *irreducible* if $R(s)$ and $N_R(s)$ are relatively right prime.

A useful concept in transfer–function matrix descriptions is the notion of a *column-proper* or *column-reduced* polynomial matrix.

Definition 9.2 An $(n_1 \times n_2)$ polynomial matrix $P(s)$ is said to be column proper (column reduced) if and only if $\Gamma_c[P(s)]$, defined in the footnote below*, is of full rank equal to min $\{n_1, n_2\}$.

In fact, any polynomial matrix can be made column proper by using elementary column (or row) operations to successively reduce the degrees of the individual columns until a column proper (column reduced) matrix is obtained.

Lemma 9.1 *If $N_R(s)$ is column proper, then the transfer–function matrix $T(s) = R(s)N_R^{-1}(s)$ is strictly proper (proper) if and only if the degree of each column of $R(s)$ is less than (less than or equal to) the degree of each corresponding column of $N_R(s)$.*

Proof See Kailath [K4] p. 385.

* The degree of a $(n_1 \times n_2)$ polynomial matrix $P(s)$ is defined as the maximum degree of all its q-order minors where $q = \min(n_1, n_2)$. In particular, the degree of the *ith column of* $P(s)$, denoted by $\partial_{ci}[P(s)]$, is defined as the degree of the element of highest degree in the ith column of $P(s)$. $\Gamma_c[P(s)]$, or simply Γ_c, denotes the matrix of scalar elements consisting of the coefficients of the highest degree s terms in each column of $P(s)$.

9.3 LINEAR PARTIAL STATE FEEDBACK

In Section 1.5, we examined the effect of linear state feedback on the closed-loop performance of linear state–space systems. A similar analysis is now undertaken for the more general differential operator representation (9.11) and (9.12). Given that this representation is controllable in the sense of Theorem 9.2, it is known that the r column degrees d_i of $N_R(D)$ are the controllability indices of the system where $\Sigma_{i=1}^{r} d_i = n$.

Consider the application of the linear partial state control law

$$u(t) = F(D)z(t) + v(t) \tag{9.13}$$

to the minimal differential operator realization (9.11) and (9.12) as in Fig. 9.1, resulting in the closed-loop differential operator representation

$$[N_R(D) - F(D)]z(t) \triangleq N_{FR}(D)z(t) = v(t) \tag{9.14}$$

$$y(t) = R(D)z(t). \tag{9.15}$$

The corresponding closed-loop system transfer–function matrix is given by

$$T_F(s) = R(s)N_{FR}^{-1}(s). \tag{9.16}$$

The *poles* of the irreducible transfer–function matrix $T(s)$ in (9.10) are the zeros of the characteristic polynomial $\Delta(s) \triangleq \det N_R(s)$.

If $T(s)$ is square and non-singular, the *zeros* of the irreducible transfer–function matrix are the roots of det $R(s)$. For general non-square systems the zeros are these frequencies s at which the rank of $R(s)$ drops below its normal rank. In fact non-square systems commonly have no zeros (cf. Section 8.3) since it is unlikely that all minors of size less than or equal to the normal rank will be simultaneously zero.

One additional advantage of the differential operator representation (9.11) and (9.12) over state–space descriptions is the immediacy in which it displays the precise effects of linear state feedback. It is directly observed from (9.10) and (9.16) that the numerator matrix $R(s)$ and hence the system zeros are unaffected by the linear partial state feedback low (9.13), unless, of course, they are cancelled by the closed-loop poles (zeros of det $N_{FR}(s)$). Furthermore, the denominator matrix $N_{FR}(s)$ remains column proper and of the same column degree as $N_R(s)$. Only the polynomial components of $N_R(s)$ of lower column degree than d_i in each column and hence the system poles are altered by linear partial state feedback. Recalling that the polynomial system (9.11) and (9.12) is always controllable, these observations are summarized in the following theorem which may be viewed as a natural extension of Theorem 1.15 and Theorem 8.2 to polynomial system models.

Theorem 9.4 *The polynomial gain matrix $F(D)$ of the linear partial state feedback control law (9.13) exists such that the set of poles of the closed-loop system (9.14) can be arbitrarily assigned.*

Specifically, the closed-loop poles, which are equal to the zeros of

$$\det N_{FR}(D) = \Delta(D) \tag{9.17}$$

where

$$\Delta(D) \triangleq D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

is the characteristic polynomial associated with an arbitrary desired set of eigenvalues, can be assigned by means of the polynomial gain matrix

$$F(D) = N_R(D) - N_{FR}(D) \tag{9.18}$$

Unless they are cancelled by the poles of the closed-loop system, the zeros of the polynomial system (9.11) and (9.12) are invariant with respect to the application of any such feedback control law (9.13) and (9.18).

Indeed, an important application of Theorem 9.4 is system zero cancellation by partial state feedback. It was noted below Theorem 9.2 that the unobservable system modes are the zeros of det $G_R(D)$. Since, by the definition given in the footnote on p. 181, $G_R(D)$ is any greatest common devisor of $N_R(D)$ and $R(D)$ in (9.11) and (9.12), the numerator matrix $R(D)$ satisfies

$$R(D) = \bar{R}(D)G_R(D) \tag{9.19}$$

for some matrix $\bar{R}(D)$. If, in the manner of Theorem 9.4, one sets

$$N_{FR}(D) = \bar{N}(D)G_R(D) \tag{9.20}$$

one has, in view of (9.19), (9.20) and (9.16), that

$$\begin{aligned} T_F(s) &= R(s)N_{FR}^{-1}(s) \\ &= \bar{R}(s)G_R(s)G_R^{-1}(s)\bar{N}^{-1}(s) = \bar{R}(s)\bar{N}^{-1}(s) \end{aligned} \tag{9.21}$$

The cancellation of all the system zeros, represented by $G_R(s)$ in (9.21), is a multivariable generalization of the well-known scalar technique of "shifting a pole under a zero" to render a system mode unobservable. It is important that these system zeros are located in the stable region of the complex plane.

9.4 OBSERVER-BASED FEEDBACK COMPENSATION

Where only certain linear combinations of the partial state are available in the observations $y(t) = R(D)z(t)$ of (9.12), it is necessary to reconstruct the partial state in order to implement the control law (9.13). Our intuition, developed in

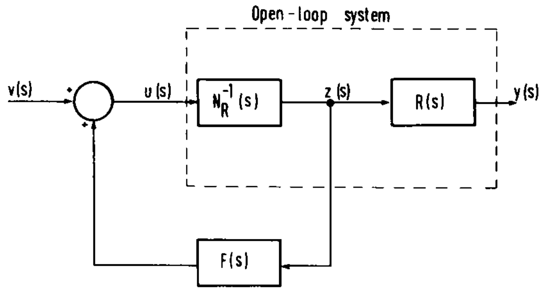

Fig. 9.1 Closed-loop partial state feedback system.

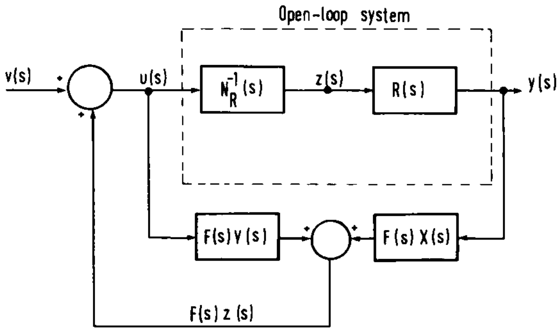

Fig. 9.2 Observer-based feedback compensator.

preceding chapters, suggests that the partial state may be generated by an observer which operates dynamically on the available system inputs and outputs. Thus, given an observable polynomial system (9.11) and (9.12) one has from (9.7b) of Theorem 9.2* that, at least in principle, the partial state $z(t) \in R^q$ may be generated dynamically from the input $u(t)$ and output $y(t)$ as

$$z(t) = X(D)y(t) + Y(D)u(t) \tag{9.22}$$

where $X(D)$ and $Y(D)$ are polynomial matrices in the differential operator D. A schematic of this observer-type feedback compensator is presented in Fig. 9.1, and should be compared with the accessible partial state feedback arrangement of Fig. 9.1.

Unfortunately, the configuration of Fig. 9.2 is not physically realizable. In order to make this configuration a realizable one, we act in accordance with the dictates of Lemma 9.1. First of all, we introduce a denominator polynomial

* While Theorem 9.2 is stated for the more general system description (9.1), subsequent analysis and notation refers to the system (9.11) and (9.12).

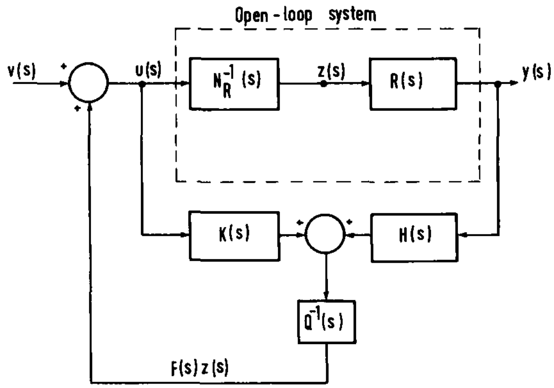

Fig. 9.3 Physically realizable observer-based compensator.

matrix, $Q(D)$ say, so that, by (9.7b) or in the notation of this section $XR + YN_R = I$,

$$[Q(D)F(D)X(D)]R(D) + [Q(D)F(D)Y(D)]N_R(D) = Q(D)F(D). \quad (9.23)$$

Secondly, we reduce the numerators $Q(D)F(D)X(D)$ and $Q(D)F(D)Y(D)$ by elementary row operations to $K(D)$ and $H(D)$ respectively so that $Q^{-1}(s)K(s)$ and $Q^{-1}(s)H(s)$ are proper (stable) transfer–function matrices. Then, a physically realizable observer-based compensator, depicted in Fig. 9.3, ensues.

In more detail, the reduction procedure is as follows. Determine a relatively left prime left factorization of $T(s)$, say,

$$T(s) = A^{-1}(s)B(s), \quad A(s) \text{ row reduced} \quad (9.24a)$$

so that

$$B(s)N_R(s) = A(s)R(s). \quad (9.24b)$$

Equations (9.23) and (9.24) may be arranged in the compact form

$$\begin{bmatrix} C(s) & E(s) \\ B(s) & -A(s) \end{bmatrix} \begin{bmatrix} N_R(s) \\ R(s) \end{bmatrix} = \begin{bmatrix} Q(s)F(s) \\ 0 \end{bmatrix} \quad (9.25)$$

where $C(s) \triangleq Q(s)F(s)Y(s)$ and $E(s) \triangleq Q(s)F(s)X(s)$. Furthermore, elementary row operations can be used to reduce (9.25) to the form

$$\begin{bmatrix} K(s) & H(s) \\ B(s) & -A(s) \end{bmatrix} \begin{bmatrix} N_R(s) \\ R(s) \end{bmatrix} = \begin{bmatrix} Q(s)F(s) \\ 0 \end{bmatrix} \quad (9.26)$$

where $H(s)$ is such that

$$\text{column degrees of } H(s) < \text{column degrees of } A(s). \quad (9.27)$$

The denominator polynomial matrix $Q(s)$ is arbitrary and can be chosen as

$$Q(s) = \begin{bmatrix} s^{\nu-1} & 0 & \dots & 0 & \alpha_1(s) \\ -1 & s^{\nu-1} & \dots & 0 & \alpha_2(s) \\ 0 & -1 & & & \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & s^{\nu-1}+\alpha_m(s) \end{bmatrix} \tag{9.28}$$

so that

$$\det Q(s) = s^{m(\nu-1)} + \alpha_m(s)s^{(m-1)(\nu-1)} + \dots + \alpha_1(s) \tag{9.29}$$

is an arbitrary (stable) polynomial of degree $m(\nu - 1)$ for any appropriate choice of $\alpha_i(s)$, $i = 1, \dots, m$, of degree less than $\nu - 1$. By Theorem 9.1, ν is the observability index associated with any state–space system equivalent to (9.11) and (9.12).

It remains to show that $K(s)$ and $H(s)$ are of column or row degree less than or equal to $\nu - 1$. Since $A^{-1}(s)B(s)$ in (9.24a) is a minimal left factorization and $A(s)$ is row reduced, the highest row or column degree of $A(s)$ is ν. Therefore, by (9.27), the highest column degree of $H(s)$ is at most $\nu - 1$.

Also, rewriting the first relation of (9.26), using (9.21), as

$$[K(s) - Q(s)]N_R(s) + H(s)R(s) = -Q(s)N_{FR}(s) \tag{9.30}$$

one has

$$[Q^{-1}(s)K(s) - I]N_R(s)N_{FR}^{-1}(s) + [Q^{-1}(s)H(s)][R(s)N_{FR}^{-1}(s)] = -I. \tag{9.31}$$

Since $Q^{-1}(s)H(s)$ is proper and $R(s)N_{FR}^{-1}(s)$ is strictly proper, the last two terms on the left-hand side at $s = \infty$ may be ignored. As remarked in Section 9.3, $N_{FR}(s)$ is a column proper and is of the same column degree as $N_R(s)$; thus $N_R(s)N_{FR}^{-1}(s)$ is proper and as a consequence $Q^{-1}(s)K(s)$ in (9.28) is proper. A physically realizable compensator, corresponding to the scheme of Fig. 9.3, is summarized in the following theorem.

Theorem 9.5 *Given the relatively right prime differential operator system (9.11) and (9.12), with $N_R(D)$ column proper, and any $F(D)$ of lower column degree than $N_R(D)$, then there exists a triple $\{Q(D), H(D), K(D)\}$ of polynomial matrices which satisfy the three conditions:*

(i) $K(D)N_R(D) + H(D)R(D) = Q(D)F(D)$, (9.32)

(ii) $Q^{-1}(s)H(s)$ and $Q^{-1}(s)K(s)$ are stable proper transfer–function matrices,

(iii) the zeros of $\det Q(s)$ *lie in the stable half-plane.*

Implicit in the preceding analysis, for instance in the derivation of Equation (9.30), is the requirement that the observer-based control scheme of Fig. 9.3 yields the *same* closed-loop transfer–function matrix as the accessible partial state feedback arrangement of Fig. 9.1. This requirement, a polynomial matrix generalization of that encountered in Section 4.5 and Section 8.6 for state–space systems, is readily confirmed in the light of Theorem 9.5 and Fig. 9.3. That is, equating signals at the first summation, we have

$$u(s) = N_R(s)z(s) = v(s) + Q^{-1}(s)[K(s)N_R(s) + H(s)R(s)]z(s) \qquad (9.33)$$

whereupon, after substitution of (9.32) one obtains after some manipulations

$$\begin{aligned} y(s) &= R(s)[N_R(s) - F(s)]^{-1}Q^{-1}(s)Q(s)v(s) \\ &= R(s)N_{FR}^{-1}(s)v(s) \end{aligned} \qquad (9.34)$$

which is precisely the desired closed-loop transfer relation under direct state feedback of (9.16) where $F(s)$ is defined by (9.18) and stable pole–zero cancellations represented by $Q(s)$ and its inverse have been made. It is noted (cf. Section 4.5) that this pole–zero cancellation is permissible in view of the assumption that det $Q(s)$ of (9.32) constitutes a stable polynomial.

In summary, the feedback compensation scheme of Theorem 9.5 and Fig. 9.3 is based on an irreducible matrix fraction transfer function or equivalently a minimal differential operator model description of the open-loop system. The physically realizable compensator takes the form

$$F(s)z(s) = Q^{-1}(s)[K(s)u(s) + H(s)y(s)] \qquad (9.35)$$

and is of dynamic order equal to the degree of det $Q(s) = r(\nu - 1)$. Although the compensation scheme requires no knowledge of state–space techniques, it is equivalent by Theorem 9.1 to a state–space system with observability index ν. In this light it may be regarded as a frequency–domain counterpart of the linear function observer of Chapter 3 where it is noted that, in general, $r(\nu - 1)$ is not the minimal order obtainable. Only in the special single-input system case ($r = 1$) is arbitrary pole assignment achieved with a compensator of minimal order $\nu - 1$ (cf. Theorem 3.1).

9.5 NOTES AND REFERENCES

The application of polynomial matrix methods [G1], [M1] to linear control systems was first undertaken by Popov [P9] and Rosenbrock [R10]. Another important and lucid exposition is contributed by Wolovich [W10] while the text [K4] of Kailath and the special issue [S2] contain much additional material and many more recent references. These methods essentially exploit the fact [G1], [M1] that properties of matrices defined over the real field $\mathscr{R}$

largely carry over to matrices of polynomials defined over any arbitrary ring $\mathscr{R}[s]$. Their significance as regards control system studies, in particular that of the polynomial system matrix, rests in the provision of all the information necessary to describe a dynamical system. In addition to state–space systems, physical systems which consist of a mixed set of algebraic and differential equations, sometimes called descriptor systems, are of this form. An analogous polynomial system approach, appropriate to discrete-time or sampled-data systems, may also be developed using the z-transform [R10].

Most of the material of Section 9.2 on polynomial system properties is drawn from the fundamental studies of Rosenbrock [R10]; other interpretations, generalizations and supplementary results are to be found in Wolovich [W10] and Kailath [K4]. Definition 9.1 is an instance of the maxim that a good theorem often becomes a definition (see Rosenbrock [R10] p. 52 and Wolovich [W11]). The important notion of a column-proper polynomial matrix treated in Definition 9.2 and Lemma 9.1 is due to Wolovich [W10]; see also Heymann [H7] and Kailath [K4] who use the perhaps more suggestive name column-reduced.

The multivariable transfer–function or differential-operator equivalent of linear state feedback and observer-based feedback, pursued in Section 9.3 and Section 9.4, is due to Wolovich [W10], [W11]. The exposition of Section 9.4 has also benefited considerably from the alternative treatment of Kailath [K4]. Earlier transfer–function feedback controllers of this genus have been obtained for single-input single-output systems by Mortensen [M20], Shipley [S10] and Chen [C2]. Although these design methods are independent of state–space procedures, it is clear that their successful derivation is due in no small measure to the prior development of the intimately related time–domain concepts of state, controllability and observability. A geometric interpretation of this time–frequency domain relationship, together with improved bounds on the minimal order of the observer, is provided by Kimura [K23]. The feedback compensation scheme of Theorem 9.5 can be generalized [W10], [W12] to include feedforward input dynamics. This extra compensation flexibility is useful in meeting other design specifications such as dynamic decoupling and exact model matching. An application of these methods to adaptive observer-based controller design is presented by Elliott and Wolovich [E1] while a host of other extensions and a wider perspective are to be found in the special issue [S2].

Chapter 10

Geometric Theory of Observers

10.1 INTRODUCTION

Like the polynomial matrix formulation of Chapter 9, geometric state–space theory is foremost a theory of system *synthesis*. Within the framework of abstract (geometric) linear algebra, it seeks to establish the qualitative structural properties of a system. For instance, in observer synthesis it addresses the questions of whether one can achieve asymptotic state (state function) reconstruction, disturbance rejection and robustness in the face of parameter variations. Only when these structural issues have been satisfactorily resolved, can one then proceed to *design* or the appropriate adjustment of parameters, within the system synthesis, to meet quantitative design requirements related to transient response, stability margins, etc. Of course, the methods of earlier chapters have been partly synthesis and partly design, and we have not always made the distinction.

The power of geometric state–space theory resides in its ability to characterize concisely the various synthesis possibilities of a system in terms of a few basic system concepts. These basis system concepts are the familiar notions of controllability and observability exhibited more abstractly as "controllable subspace" and "unobservable subspace" of the state–space. A rudimentary geometric theory would go something like this.

Consider the linear state–space system* $S(A, B, C)$. The interaction of the control inputs on the system modes depends on $\mathscr{B}$ = image B. Those system modes which are susceptible to control action lie in a controllable subspace, designated $\mathscr{R} = \langle A \mid \mathscr{B} \rangle$, which is the smallest A-invariant subspace of the state–space containing $\mathscr{B}$. Conversely those system modes which are not observed at the output lie in kernel C. In fact, the largest A-invariant subspace

* All geometric considerations apply equally to continuous and discrete state–space systems: henceforth, we treat only continuous-time models $S(A, B, C)$. In so doing we waive any distinction between controllability and reachability.

of the state–space contained in kernel C is the unobservable subspace $\mathcal{N}$. An A-invariant subspace, by the way, is any subspace $\mathcal{V}$ satisfying $A\mathcal{V} \subset \mathcal{V}$. An observer, then, is essentially a dynamic device which lives in Ker C. It supplies the part of the system state that the system outputs cannot directly tell us about. Just as transmission-blocking in the frequency-domain (Section 8.3) depends on $\mathcal{B} = \text{Im } B$ and Ker C, so also in the time–domain the control action does not influence the output if

$$\langle A \mid \mathcal{B} \rangle \subset \text{Ker } C.$$

A more sustained exposure to geometric state–space ideas is to be had in Section 10.2. Section 10.3 and Section 10.4 establish the synthesis of minimal-order state observers and linear state function observers, respectively, in terms of distinguished subspaces. Emphasis throughout is on the translation of subspace algegra into matrix algebra and vice versa. Not only has this a great pedagogical value in building upon earlier less abstract material, but it also forms an indispensable part of any computational solution of the synthesis. We end on a less abstract note in Section 10.5 when we address the robustness of observers to small parameter variations: a very important synthesis question and one that, fortunately, can be resolved in the context of observer-based controller synthesis.

10.2 PRELIMINARY DEFINITIONS AND CONCEPTS

Consider the finite-dimensional linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{10.1}$$

$$y(t) = Cx(t) \tag{10.2}$$

where $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$ and $y(t) \in \mathcal{Y}$. It is recalled from Section 1.3 that if dimension (dim) $\mathcal{X} = n$ the system (10.1) and (10.2) is completely observable if and only if the $(n \times nm)$ matrix

$$Q = [C', A'C', \dots, A'^{n-1}C'] \tag{10.3}$$

is of full rank n. Interpreted more abstractly, noting that rank Q + nullity Q $= n$, we have the following equivalent definition of observability.

Definition 10.1 The pair of maps (A, C) is completely observable if

$$\bigcap_{i=1}^{n} \text{Ker } (CA^{i-1}) = 0. \tag{10.4}$$

Of course, we know from the canonical decomposition theorem, Theorem

1.11, that in general a system will consist of four subsystems of which only one subsystem is both completely controllable and completely observable. In geometric terms, the whole state space $\mathscr{X}$ decomposes into four independent subspaces in accordance with

$$\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3 \oplus \mathscr{X}_4 \tag{10.5}$$

where

$$\begin{gathered} \mathscr{X}_1 = \mathscr{N} \cap \mathscr{R}, \quad \mathscr{X}_1 \oplus \mathscr{X}_2 = \mathscr{R} \\ \mathscr{X}_1 \oplus \mathscr{X}_3 = \mathscr{N}, \quad (\mathscr{N} + \mathscr{R}) \oplus \mathscr{X}_4 = \mathscr{X} \end{gathered} \tag{10.6}$$

and $\mathscr{N}$ and $\mathscr{R}$ are the unobservable subspace and controllable subspace respectively. It is clear from (10.5) and (10.6), that $\mathscr{X}_1$ represents the "controllable but unobservable" subspace, $\mathscr{X}_2$ represents the "controllable and observable" subspace, etc.

Subsequently, our attention is confined in the main to the question of observability of these subsystems and their associated subspaces. In fact, one has a natural corollary to Definition 10.1 in the complementary notion of an *unobservable subspace* alluded to above.

Definition 10.2 The unobservable subspace $\mathscr{N} \subset \mathscr{X}$ of the pair (A,C) is defined as

$$\mathscr{N} = \bigcap_{i=1}^{n} \operatorname{Ker}(CA^{i-1}). \tag{10.7}$$

It is noted, in particular, that $A\mathscr{N} \subset \mathscr{N}$; that is, $\mathscr{N}$ is an *A-invariant subspace.* Indeed, $\mathscr{N}$ is the largest A-invariant subspace contained in $\operatorname{Ker} C$; every $\tilde{x} \in \mathscr{N}$ gives rise to a zero output $y = C\tilde{x}$.

Just as in earlier chapters, one might usefully eliminate the unobservable states to obtain a lower order system that is completely observable, one can equivalently factor out the unobservable subspace $\mathscr{N}$ from the original state–space $\mathscr{X}$ to be left with a factor space $\mathscr{X}/\mathscr{N}$ of lower dimension, $\dim \mathscr{X} - \dim \mathscr{N}$, that is observable. Denoting this observable factor space by $\bar{\mathscr{X}} = \mathscr{X}/\mathscr{N}$, let $P: \mathscr{X} \to \bar{\mathscr{X}}$ be the corresponding canonical projection and $\bar{A}: \bar{\mathscr{X}} \to \bar{\mathscr{X}}$ be the corresponding map induced in $\bar{\mathscr{X}}$ by A. Since $\operatorname{Ker} C \supset \mathscr{N}$, there exists a map $\bar{C}: \bar{\mathscr{X}} \to \mathscr{Y}$ such that $\bar{C}P = C$, as shown below.

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "A"] \arrow[dd, "P"'] & \mathscr{X} \arrow[dd, "P"] \arrow[dr, "C"] & \\
& & \mathscr{Y} \\
\bar{\mathscr{X}} \arrow[r, "\bar{A}"] & \bar{\mathscr{X}} \arrow[ur, "\bar{C}"'] &
\end{tikzcd} \tag{10.8}$$

Lemma 10.1 *The pair $(\bar{A}, \bar{C})$ is observable.*

Proof Since $n = \dim(\mathscr{X}) \geqslant \dim(\bar{\mathscr{X}})$ it is enough to show that

$$\bar{\mathscr{N}} = \bigcap_{i=1}^{n} \operatorname{Ker}(\bar{C}\bar{A}^{i-1}) = 0.$$

If $\bar{x} = Px \in \bar{\mathscr{N}}$, then $\bar{C}\bar{A}^{i-1}Px = 0$, $i = 1, \ldots, n$. From the commutative diagram above there results $CA^{i-1}\bar{x} = 0$, $i = 1, \ldots, n$, i.e. $x \in \mathscr{N}$, so $\bar{x} = Px = 0$. Q.E.D. □

Since $(\bar{A}, \bar{C})$ is an observable pair, it is always possible to construct an observer to estimate the coset of the system state *modulo* the unobservable subspace, $\bar{x} \in \bar{\mathscr{X}} = \mathscr{X}/\mathscr{N}$, of the factor system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}\bar{x}. \tag{10.9}$$

Instead of working with an observable system pair (A, C), it is sometimes necessary, and often convenient, to analyse the dual controllable pair (A, B) or (A', C'). The dual *controllable subspace* $\mathscr{R}$ is described by

$$\langle A \,|\, \mathscr{B} \rangle \triangleq \mathscr{B} + A\mathscr{B} + \cdots + A^{n-1}\mathscr{B} \tag{10.10}$$

where $\mathscr{B} = \operatorname{Im} B$. From (10.10) it is clear that $A\mathscr{R} \subset \mathscr{R}$; indeed, $\mathscr{R}$ is the smallest A-invariant subspace containing $\mathscr{B}$. A geometric version of Theorem 1.3 ensues.

Lemma 10.2 *The pair (A, B) is completely controllable if*

$$\langle A \,|\, \mathscr{B} \rangle = \mathscr{X}. \tag{10.11}$$

The uncontrollable subspace, the subspace $\bar{\bar{\mathscr{X}}}$ which cannot be reached by any control $u(\cdot) \in \mathscr{U}$, of any general system decomposition (10.5) is defined in dual fashion as $\bar{\bar{\mathscr{X}}} = \mathscr{X} \,|\, \mathscr{R}$ where $\mathscr{R}$ is the controllable subspace.

Lemma 10.3 *The pair (A, C) is observable if and only if the pair (A', C') is controllable.*

Proof One has

$$\mathscr{N}^{\perp} = \left[\bigcap_{i=1}^{n} \operatorname{Ker}(CA^{i-1})\right]^{\perp} = \sum_{i=1}^{n} [\operatorname{Ker}(CA^{i-1})]^{\perp}$$
$$= \sum_{i=1}^{n} \operatorname{Im}(A'^{i-1}C') = \langle A' \,|\, \operatorname{Im} C' \rangle$$

and therefore $\mathscr{N} = 0$ if and only if $\langle A' \,|\, \operatorname{Im} C' \rangle = \mathscr{X}'$. Q.E.D. □

The *restriction of a map* A to a subspace $\mathscr{R} \subset \mathscr{X}$ is denoted by $A \mid \mathscr{R}$; that is $A \mid \mathscr{R}$ has the action of A on $\mathscr{R}$ but is not defined off $\mathscr{R}$. The spectrum of such a restricted map is denoted by $\lambda[A \mid \mathscr{R}]$. Finally, we describe the subspace $\mathscr{U}$ as (A, B)-*invariant* if there exists a map $F: \mathscr{X} \to \mathscr{U}$ such that

$$(A + BF)\mathscr{V} \subset \mathscr{V}. \tag{10.12}$$

Namely, if $x(\cdot)$ starts in $\mathscr{V}$, it stays in $\mathscr{V}$ under the action of $u = Fx$.

10.3 MINIMAL-ORDER STATE OBSERVERS

Given m linear combinations of the state $x(t) \in R^n$ of the observable system (10.1) in the measurements (10.2), a dynamic observer of minimal order $n - m$ can be constructed to estimate the remaining $n - m$ linear state combinations on the basis of the available system inputs and outputs. In the manner of Chapter 1, an algebraic formulation of the state observation solution is as follows.

Let Λ be a set of $n - m$ numbers of which complex entries only occur in conjugate pairs. There exist matrices $V \in R^{(n-m)n}$, $K \in R^{nm}$ and $D \in R^{(n-m) \times (n-m)}$ such that

$$\operatorname{rank}\begin{bmatrix} C \\ V \end{bmatrix} = n \tag{10.13}$$

$$V(A - KC) = DV$$

or

$$(A - KC)'V' = V'D' \tag{10.14}$$

$$\lambda(D) = \Lambda. \tag{10.15}$$

The associated minimal-order observer† is described by

$$\dot{z}(t) = Dz(t) + VKy(t) + VBu(t) \tag{10.16}$$

$$x(t) \to \begin{bmatrix} C \\ V \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}. \tag{10.17}$$

It is illuminating to interpret the state observation solution (10.13) to (10.15) from a geometric point of view.

Theorem 10.1 *Let (A, C) be an observable pair $\mathscr{C}' = \operatorname{Im} C'$ and $\dim \mathscr{C}' = m$. Let Λ be a set of $n - m$ self-conjugate numbers. There exist an $(n - m)$-dimensional subspace $\mathscr{V}' \subset \mathscr{X}'$, and a map $K: \mathscr{Y} \to \mathscr{X}$ such that*

† The notation of this chapter is quite different from that adopted in Chapter 1 and Chapter 2. However, the observer (10.16) is identical to (1.86) if in the notation of Chapter 1 one sets $K = AV$.

$$\mathscr{C}' \oplus \mathscr{V}' = \mathscr{X}' \tag{10.18}$$

$$(A - KC)'\mathscr{V}' \subset \mathscr{V}' \tag{10.19}$$

$$\lambda[(A - KC)' \,|\, \mathscr{V}'] = \Lambda. \tag{10.20}$$

The proof of Theorem 10.1, based on the controllability of the dual pair (A', C'), is to be found in Wonham [W16], [W17]. Instead, we offer a few remarks on the geometric significance of the state observation solution (10.18) to (10.20). Equation (10.18) is a statement in the dual state space $\mathscr{X}'$ to the effect that

$$\begin{bmatrix} C \\ V \end{bmatrix} : \mathscr{X} \to \mathscr{Y} \oplus \mathscr{V}. \tag{10.21}$$

In other words, the entire state–space $\mathscr{X}$ may be asymptotically reconstructed by complementing the coset of $x(t)$ in the m-dimensional quotient space $\mathscr{X} \,|\, \text{Ker } C$ by the component $x(t)$ in the $(n - m)$-dimensional subspace Ker C. It is noted that $\mathscr{V}'$ in (10.19) is an (A', C')-invariant subspace in the sense of (10.12); that is, any $v' \in \mathscr{V}'$ remains in $\mathscr{V}'$ under the operation of $(A - KC)'$. Moreover, we have in (10.20) that the eigenvalues of the map $(A - KC)'$, restricted to the $(n - m)$-dimensional subspace $\mathscr{V}' \subset \mathscr{X}'$, can be arbitrarily assigned to any desired set Λ. In a full-order observer synthesis, this map restriction is waived and $\mathscr{V}' = \mathscr{X}'$.

10.3.1 Detectability and detectors

A weaker structural condition on the system (10.1) and (10.2) than that of complete observability is that of detectability or observability of the unstable system modes. Since the minimum polynomial of A is the monic polynomial $\alpha(\lambda)$ of least degree such that $\alpha(A) = 0$, it may be factored as the product

$$\alpha(\lambda) = \alpha^+(\lambda)\bar{\alpha}(\lambda)$$

where the zeros of $\alpha^+(\alpha^-)$ over $\mathbb{C}$ belong to the closed right (open left) half plane and

$$\mathscr{X}^+(A) \triangleq \text{Ker } \alpha^+(A), \quad \mathscr{X}^-(A) \triangleq \text{Ker } \alpha^-(A).$$

Definition 10.3 The pair (A, C) is detectable if

$$\bigcap_{i=1}^{n} \text{Ker}\,(CA^{i-1}) \subset \mathscr{X}^-(A). \tag{10.22}$$

In words, the system (10.1) and (10.2) is detectable if the unobservable subspace of Definition 10.2 is a proper subspace of the subspace $\mathscr{X}^-(A)$ of stable modes of A.

If it is only required to stabilize the system, it is sufficient that an observer estimate the unstable system modes or x *modulo* the subspace $\mathscr{X}^-(A)$ of stable modes of A. Such a restricted observer, or *detector*, exists provided that the pair (A, C) is detectable and is in general of lower dynamic order than dim (Ker C).

A dynamic model that (at least) describes the unstable subsystem is given by

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{10.23}$$

$$\bar{y} = \bar{C}\bar{x}$$

where $\bar{x}(t) \in \bar{\mathscr{X}}$ and $\bar{\mathscr{X}}$ is the factor space $\mathscr{X} | \mathscr{C}$,

$$A\mathscr{C} \subset \mathscr{C} \subset \mathscr{X}^-(A).$$

Having arranged that $(\bar{A}, \bar{C})$ is an observable pair, the minimal detector problem is one of synthesizing a state observer of minimal order that will estimate the state $\bar{x} \in \bar{\mathscr{X}}$ of the subsystem (10.23) on the basis of the system input u and output $\bar{y} = Dy$ where $\bar{y}$ contains no more information than y.

10.3.2 Observability under state feedback

As we have seen in the preceding discussion, it is imperative that the unobservable system modes be stable. We now briefly look at the converse problem of making the stabilized system modes unobservable. Although the controllability of a system is invariant under state feedback, the observability of such a system is not since (cf. Section 9.3) state feedback can be used to eliminate the zeros of the system through pole–zero cancellation. Thus, having stabilized the unstable system modes, it is possible to use the design freedom beyond arbitrary pole assignment to render as many as possible of these stabilized modes unobservable. Through making these stabilized modes unobservable, an observer of reduced dynamic order need only be employed to reconstruct the remaining observable system modes. Furthermore, the observability of the reduced-order observable subsystem will be unaffected by linear feedback since, in geometric terms, one will have already factored out the (maximal) unobservability subspace.

It is recalled from Definition 10.2 that the unobservable subspace $\mathscr{N}$ of the pair (A, C) is the largest A-invariant subspace contained in Ker C. Thus, the unobservable subspace v^* of the pair of interest, namely $(A + BF, C)$, is the subspace of largest possible dimension such that

$$(A + BF)v^* \subset v^*, \quad v^* \subset \operatorname{Ker} C.$$

Also, we note from (10.12) that v^* is a unique (A, B)-invariant subspace known as a *supremal* or *largest* (A, B)-invariant subspace in that no other (A, B)-invariant subspace is of greater dimension.

10.4 LINEAR FUNCTION OBSERVERS

Consider the linear finite-dimensional system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bd(t)\\ y(t) &= Cx(t)\\ r(t) &= Hd(t)\end{aligned} \tag{10.24}$$

where $x(t)\in\mathscr{X}$ is the state, $d(t)\in\mathscr{D}$ is a disturbance, $y(t)\in\mathscr{Y}$ is the observation of the state and $r(t)\in\mathscr{R}$ is the observation of the disturbance.

It is required to synthesize an observer in order to asymptotically reconstruct the linear state function

$$w(t) = Dx(t) \tag{10.25}$$

An estimate of the linear state function $w(t)$ is generated by the dynamical observer

$$\begin{aligned}\dot{z}(t) &= Nz(t) + My(t) + Lr(t)\\ \hat{w}(t) &= Sz(t) + Ty(t)\end{aligned} \tag{10.26}$$

where $z(t)\in\mathscr{Z}$. Equations (10.24) and (10.26) together define a composite system in the state-space $\mathscr{X}\oplus\mathscr{Z}$

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = \begin{bmatrix}A & 0\\ MC & N\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix} + \begin{bmatrix}B\\ LH\end{bmatrix}d(t) \tag{10.27}$$

$$e(t) \triangleq w(t) - \hat{w}(t) = [D - TC, -S]\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}$$

where $e(t)\in\mathscr{Z}$ is the observer error. Let the composite system matrix be

$$A_c = \begin{bmatrix}A & 0\\ MC & N\end{bmatrix}. \tag{10.28}$$

For zero observer error $e(t)$, it is required that the largest A_c-invariant subspace be contained in Ker $[D - TC, -S]$ or

$$\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}$$

be unobservable from $e(t)$ in the sense of Definition 10.2. This observer errorless subspace, denoted by $\mathscr{V}$, contains the initial composite state

$$\begin{bmatrix}x(0)\\ z(0)\end{bmatrix}$$

from which the composite system will move in such a way that the error is zero for all time. $\mathscr{V}$ is naturally related to two subspaces of the original state space $\mathscr{X}$, which are denoted by $\mathscr{V}_i$ and $\mathscr{V}_p$ (i for insertion, p for projection).

$$\mathscr{V}_i = \left\{ x \in \mathscr{X} \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathscr{V} \right\} \tag{10.29}$$

$$\mathscr{V}_p = \left\{ x \in \mathscr{X} \,|\, \exists z \in \mathscr{Z} : \begin{bmatrix} x \\ z \end{bmatrix} \in \mathscr{V} \right\}. \tag{10.30}$$

In order that the asymptotic reconstruction of $w(t)$ proceeds satisfactorily, it is crucial that the composite system-observer (10.27) be stable. The stability of (10.27) is determined by the nature of the system disturbance $d(t)$ and the eigenvalues of A_c. If the disturbance $d(t)$ is known to possess a dynamic structure it may be modelled by a state equation which is incorporated in the system description (10.24). An observer may then be synthesized for the augmented system in the usual fashion (Johnson [J2], O'Reilly [O6]). In the present instance, since nothing is known about the dynamic behaviour of $d(t)$, it is necessary that the disturbance $d(t)$ be decoupled from the error or the forced response.

$$e(t) = (D - TC, \; -S) \int_0^t e^{A_c(t-s)} \begin{bmatrix} B \\ LH \end{bmatrix} d(s)\, ds = 0 \tag{10.31}$$

for all $d(\cdot) \in D$ and $t \geqslant 0$. Equivalently, we require the controllable subspace

$$\left\langle A_c \,\middle|\, \operatorname{Im} \begin{bmatrix} B \\ LH \end{bmatrix} \right\rangle \subset \operatorname{Ker}(D - TC, \; -S) \tag{10.32}$$

or

$$\left\langle A_c \,\middle|\, \operatorname{Im} \begin{bmatrix} \mathrm{B} \\ \mathrm{L\,H} \end{bmatrix} \right\rangle \subset \mathscr{V}. \tag{10.33}$$

Then, the exponents appearing in the dynamics of the error $e(t)$ will be precisely the points in the spectrum of the map $\bar{A}_c$ induced by A_c on the factor space $(\mathscr{X} \oplus \mathscr{Z})/\mathscr{V}$. So, as our second stability requirement we have

$$\lambda(\bar{A}_c) \subset \mathbb{C}_g \tag{10.34}$$

where $\mathbb{C}_g$ is some desired region in the (stable) open left half-plane and $\mathbb{C}_b$ is its complement $\mathbb{C}/\mathbb{C}_g$. Alternatively, (10.34) may be written as

$$(\mathscr{X} \oplus \mathscr{Z})_b(A_c) \subset \mathscr{V}. \tag{10.35}$$

Minimal-order stable observer problem: Given a system (10.24), find a linear function observer of lowest possible order such that its errorless subspace $\mathscr{V}$

contains both

$$\left\langle A_c \mid \operatorname{Im}\begin{bmatrix} B \\ LH \end{bmatrix}\right\rangle \quad \text{and} \quad (\mathscr{X} \oplus \mathscr{Z})_b(A_c).$$

10.4.1 Observer synthesis

Sufficient conditions for the dynamic system (10.26) to act as an observer for the linear state function (10.25) of the system (10.24) are now presented. These conditions, described as properties of the pair of subspaces $(\mathscr{V}_i, \mathscr{V}_p)$ of the original state–space $\mathscr{X}$, are for the most part geometric analogues of those of Theorem 3.2.

Theorem 10.2 *Given the system (10.24) and (10.25), and a pair of subspaces of* $\mathscr{X}$, $(\mathscr{V}_i, \mathscr{V}_p)$, *having the following properties:*

(*i*) $\mathscr{V}_i \subset \mathscr{V}_p$, $\mathscr{V}_i$ *is* (A, C)*-invariant and* $\mathscr{V}_p$ *is* A*-invariant;*

(*ii*) $\mathscr{V}_i \cap \operatorname{Ker} C \subset \operatorname{Ker} D$;

(*iii*) $\mathscr{V}_i \supset B \operatorname{Ker} H$ *and* $\mathscr{V}_p \supset \operatorname{Im} B$;

(*iv*) *there exists* $G: \mathscr{Y} \to \mathscr{X}$ *such that* $(A + GC)\mathscr{V}_i \subset \mathscr{V}_i$, $\operatorname{Im} G \subset \mathscr{V}_i$ *and* $\lambda(\overline{A + GC}) \subset \mathbb{C}_g$, *where* $\overline{A + GC}$ *is the induced operator on* $\mathscr{X}/\mathscr{V}_i$.

Then one can construct an observer for (10.24) and (10.25) of the order $\dim \mathscr{V}_p - \dim \mathscr{V}_i$.

The possibility of $\operatorname{Ker} C \subset \operatorname{Ker} D$ is excluded since no dynamic observer is required in this case.

Proof The proof consists of showing that an observer constructed in accordance with conditions (i) to (iv) will meet the two stability requirements (10.33) and (10.35). To this end, it is helpful to interpret the geometric conditions (i) to (iv) in matrix terms.

Let $\mathscr{Z}$ be any linear space of dimension $\dim \mathscr{V}_p - \dim \mathscr{V}_i$, and let R be a linear mapping from $\mathscr{V}_p$ onto $\mathscr{Z}$ with $\operatorname{Ker} R = \mathscr{V}_i$. Write

$$\hat{\mathscr{V}} = \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} \middle| \, x \in \mathscr{V}_p \right\}.$$

Let $G: \mathscr{Y} \to \mathscr{X}$ be a map that satisfies condition (iv), take

$$M = -RG \tag{10.36}$$

and define N by

$$NRx = R(A + GC)x \quad \text{for all } x \in \mathscr{V}_p. \tag{10.37}$$

Working on condition (ii), let $\{v_1, \ldots, v_s\}$ be a basis of $\mathscr{V}_i$ such that $\{v_1, \ldots, v_s\}$

is a basis for $\mathscr{V}_i \cap \operatorname{Ker} C$. Because the elements $Cv_{s+1}, \ldots, Cv_r$ of Ker C are linearly independent, one can choose $T\colon \mathscr{Y} \to \mathscr{X}$ such that $TCv_i = Dv_i$ ($i = s + 1, \ldots, r$). Note that this relation holds automatically for $i = 1, \ldots, s$, since $TCv_i = 0 = Dv_i$ in this case. Thus we have

$$\operatorname{Ker} R = \mathscr{V}_i \subset \operatorname{Ker}(D - TC)$$

and, therefore, we can take S such that

$$SRx = (D - TC)x \quad \text{for all } x \in \mathscr{V}_p. \tag{10.38}$$

Rewriting condition (iii) as $\operatorname{Ker} H \subset \operatorname{Ker}(RB)$, L may be chosen to satisfy

$$LH = RB. \tag{10.39}$$

All the parameter matrices of the system (10.26) are specified, and it remains to show that such a constructed system satisfies the two stability requirements (10.33) and (10.35). By (10.36) and (10.37) we have that

$$(MC + NR)x = RAx \quad \text{for all } x \in \mathscr{V}_p. \tag{10.40}$$

so that

$$\begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ Rx \end{bmatrix} = \begin{bmatrix} Ax \\ RAx \end{bmatrix} \quad \text{for all } x \in \mathscr{V}_p. \tag{10.41}$$

Thus $\tilde{\mathscr{V}}$ is an A_c-invariant subspace. It follows immediately from (10.38) that $\tilde{\mathscr{V}} \subset \operatorname{Ker}(D - TC, -S)$, and from (10.39) that

$$Im \begin{bmatrix} B \\ LH \end{bmatrix} \subset \tilde{\mathscr{V}}.$$

Let $\mathscr{V}$ be the largest A_c-invariant subspace of $\operatorname{Ker}(D - TC, -S)$; then it follows that $\tilde{\mathscr{V}} \subset \mathscr{V}$ and therefore also

$$\operatorname{Im} \begin{bmatrix} B \\ LH \end{bmatrix} \subset \mathscr{V} \quad \text{or} \quad (D - TC, S) \begin{bmatrix} B \\ LH \end{bmatrix} = 0. \tag{10.42}$$

Hence the satisfaction of the first stability requirement (10.33).

With regard to the second stability requirement (10.35), note first of all that the spectrum of A_c on $(\mathscr{X} \oplus \mathscr{Z})/\mathscr{V}$ is part of the spectrum of the operator induced by A_c on $(\mathscr{X} \oplus \mathscr{Z})/\tilde{\mathscr{V}}$; so it suffices to prove that the latter spectrum, denoted by $\lambda(\bar{A}_c)$, is in $\mathbb{C}_g$. Just as we have from the block-triangular form of the map A_c on $\mathscr{X} \oplus \mathscr{Z}$ in Equation (10.28) that

$$\lambda(A_c) = \lambda(A) \cup \lambda(N) \tag{10.43}$$

it is readily deduced that on the factor space $(\mathscr{X} \oplus \mathscr{Z})/\tilde{\mathscr{V}}$

$$\lambda(\bar{A}_c) = \lambda(\bar{A}) \cup \lambda(N). \tag{10.44}$$

Since $\operatorname{Im} G \subset \mathscr{V}_p$ and $\mathscr{V}_p$ is A-invariant, it follows that $\mathscr{V}_p$ is $(A + GC)$-invariant; moreover, we have

$$\lambda(\overline{A + GC}) = \lambda(\overline{A + GC/\mathscr{V}_p}) \cup \lambda(\bar{A}). \tag{10.45}$$

But it follows from (10.37) that

$$\lambda(N) = \lambda(\overline{A + GC/\mathscr{V}_p}). \tag{10.46}$$

By (10.46) and (10.44), we get

$$\lambda(\bar{A}_c) = \lambda(\overline{A + GC}) \in \mathbb{C}_g. \quad \text{Q.E.D.} \quad \square \tag{10.47}$$

In the preceding proof it is noted that Equations (10.36), (10.37), (10.38) and (10.46) are identical to those of Theorem 3.2. Indeed, it is recalled that Theorem 3.2 supplies necessary and sufficient algebraic conditions for the existence of a linear function observer without disturbance decoupling: no system disturbance is present in the problem formulation. For the present problem, condition (iii) of Theorem 10.2, translated in matricial terms as (10.39) or (10.42), meets the additional requirement of disturbance decoupling. In the light of these observations, it seems certain that the geometric conditions of Theorem 10.2 are also *necessary* ones if an observer (10.26) is to exist for the state function reconstruction problem of (10.24) and (10.25). An abstract proof of this is presented by Shumacher [S4].

As a corollary to Theorem 10.2, we conclude with an alternative formulation of the minimal-order stable observer problem.

Corollary 10.1 *The minimal-order stable observer problem is equivalent to the problem of finding amongst all pairs of subspaces $(\mathscr{V}_i, \mathscr{V}_p)$ satisfying the requirements (i)–(iv) given in Theorem 10.2, one for which the value of* $\dim \mathscr{V}_p - \dim \mathscr{V}_i$ *is a minimum.*

10.5 ROBUST OBSERVERS

Consider again the problem of reconstructing the linear state function

$$w(t) = Dx(t) \tag{10.48}$$

for the nominal linear unforced system

$$\dot{x}(t) = A_0 x(t) \tag{10.49a}$$

$$y(t) = C_0 x(t). \tag{10.49b}$$

As we have just seen, a system of the structure

$$\dot{z}(t) = N_0 z(t) + M_0 y(t) \tag{10.50}$$

will asymptotically reconstruct (10.48) provided the parameter matrices are suitably chosen such that $z(t) - Rx(t) \to 0$ faster than the dynamics of (10.49). In view of (10.37) and (10.45), noting that $M_0 = -RG$, the system (10.50) is an observer if and only if

$$\lambda(N_0) \subset \mathbb{C}_g \tag{10.51}$$

and

$$N_0 R - RA_0 + M_0 C_0 = 0. \tag{10.52}$$

In the event of small deviations δN and δM of the observer parameter matrices N and M from their respective nominal values N_0 and M_0

$$\begin{aligned} N &\triangleq N_0 + \delta N \\ M &\triangleq M_0 + \delta M \end{aligned} \tag{10.53}$$

the open-loop nature of the observer (10.50) suggests that asymptotic state reconstruction might be seriously impaired. Indeed, if (10.50) is to continue to provide asymptotic state reconstruction in the face of such small parameter variations, it is required by (10.51) and (10.52) that

$$\lambda(N_0 + \delta N) \subset \mathbb{C}_g \tag{10.54}$$

and

$$(N_0 + \delta N)R - RA_0 + (M_0 + \delta M)C_0 = 0. \tag{10.55}$$

From (10.52) and (10.55) we have

$$[\delta M, \delta N]\begin{bmatrix} C_0 \\ R \end{bmatrix} = 0. \tag{10.56}$$

Since δM and δN are arbitrary and

$$m \leqslant \operatorname{rank}\begin{bmatrix} C_0 \\ R \end{bmatrix} \leqslant m + p$$

($\dim \mathscr{Y} = m$ and $\dim \mathscr{Z} = p$), condition (10.56) fails to hold for almost every $(\delta N, \delta M)$.

Theorem 10.3 *The system (10.50), acting as an observer for the nominal observer parameter matrices (N_0, M_0), fails to provide asymptotic state function reconstruction for almost all parameter perturbations $(\delta N, \delta M)$.*

When specialized to the case of minimal-order state observers, we have from (10.13) that

$$\operatorname{rank}\begin{bmatrix} C_0 \\ R \end{bmatrix} = n \quad (n = m + p) \tag{10.57}$$

so that, to satisfy (10.56), the elements of $(\delta N, \delta M)$ must assume zero value or, more picturesquely [B9], the parameter space $(\delta n, \delta m)$ shrinks to the origin of $\mathscr{R}^{p^2+pm}$.

Corollary 10.2 *If A_0 is completely unstable and (10.50) is a minimal-order state observer, then the state reconstruction error fails to converge for each and every perturbation $(\delta N, \delta M)$ in (N_0, M_0).*

Theorem 10.3 and Corollary 10.2 attest to the fact that conventional open-loop observer state reconstruction is highly sensitive to arbitrarily small observer parameter variations. It is proved by Bhattacharyya [B9] that if a linear dynamic system is to provide observer action despite arbitrary small parameter perturbations, it must (1) be a closed-loop system, i.e. be driven by the observer error; (2) possess redundancy, i.e. the observer must generate, implicitly or explicitly, at least one linear combination of states that is already contained in the measurements; and (3) contain a perturbation-free model of the portion of the system observable from the external input to the observer.

It is usually the case, however, that the observer (10.50) forms part of a closed-loop observer-based compensator. The results of Section 8.6 suggest that the compensation scheme may be robust with respect to small parameter variations. Specifically, consider the linear system

$$\dot{x}(t) = A_0 x(t) + B_0 u(t)$$
$$y(t) = C_0 x(t) \tag{10.58}$$

where A_0 is unstable. Suppose, as in Section 3.3, that an observer is used to realize the linear feedback control law

$$u(t) = F_0 x(t)$$

so that

$$A_0 + B_0 F_0 \text{ is a stability matrix.} \tag{10.59}$$

The observer-based controller takes the form

$$\dot{z}(t) = N_0 z(t) + M_0 y(t) + G_0 u(t) \tag{10.60}$$

$$u(t) = P_0 z(t) + V_0 y(t) \tag{10.61}$$

where the observer relations

$$P_0 R + V_0 C_0 = F_0 \tag{10.62}$$

$$G_0 = R B_0 \tag{10.63}$$

are satisfied. We finish on a happier note than Theorem 10.3 in the following theorem due to Bhattacharyya [B9], which we state without proof.

Theorem 10.4 *Given the open-loop unstable system (10.58) and the observer-based controller (10.60) to (10.63), the overall closed-loop system-observer is stable for the nominal parameters and remains stable for a class of arbitrary small perturbations* (δN, δM, δG, δP, δV, δA, δB, δC) *about the nominal.*

So despite acute susceptibility of the open-loop state function reconstruction process to arbitrarily small parameter perturbations, the overall closed-loop observer-based control system retains its stabilizing action in the face of small parameter changes.

10.6 NOTES AND REFERENCES

One of the first linear system problems to which the geometric approach was applied was the synthesis of minimal-order state observers for systems with unknown inputs by Basile and Marro [B3]; see also Guidorzi and Marro [G9] and Basile and Marro [B4]. Many of the concepts of the geometric theory such as the (A, B)-invariant subspace were developed earlier by Luenberger [L10]. The notion of an (A, B)-invariant subspace and its use in solving the disturbance decoupling problem, touched on in Section 10.4, were presented independently by Basile and Marro [B3] and by Wonham and Morse [W18]. For a definitive study of geometric state–space theory and its many applications to system problems of a synthesis nature, including observers, the elegant monograph of Wonham [W17] is essential reading. Although originally part of a time–domain formulation, the concept of (A, B)-invariant subspaces and the more advanced concept of controllability subspaces are intimately linked to the notion of frequency transmission discussed in Section 8.3; see also Karcanias and Kouvaritakis [K20]. In addition, the characterization of controllability subspaces in terms of the polynomial matrices of Chapter 9 has been stimulated by the important publication of Warren and Eckberg [W3].

Section 10.3 on minimal-order state observer theory is fashioned after Wonham [W16], [W17], while Section 10.4 on linear function observers is largely influenced by Schumacher [S4]. Closely related synthesis procedures for disturbance localization and eigenvalue assignment, employing observer-based controller and dual observer-based controllers, are recently established by Schumacher [S5] and Imai and Akashi [I6]. Further results on the minimal detector problem of Section 10.3.1 are also to be found in Schumacher [S4]. An earlier geometric solution to the scalar linear state function reconstruction problem is presented by Wonham and Morse [W18]. Originating in that work was the geometric idea of dynamic covers, and this has subsequently been developed by Kimura [K23] to establish lower and upper bounds for the

minimal order of linear function observers possessing an arbitrary set of eigenvalues. Linear function observers of mixed-type where the noisy components of the system measurements are excluded from the static part (but included in the dynamic part) of the estimation scheme are considered by Kimura [K25]. An extended account of robust observer and robust observer-based controller syntheses, treated in Section 10.5, is described in Bhattacharyya [B9]; see also Bhattacharyya [B10]. The effect of gain variation on the closed-loop poles of of a system employing observer-based feedback compensation is analysed by Friedland and Kosut [F10].

Structural properties of minimum-time linear function observers are studied by Kimura [K24]. A main result is the resolution of a long outstanding open problem, the determination of minimal order of the deadbeat observer. Further parameter insensitivity aspects of deadbeat linear function observers are treated by Akashi and Imai [A3].

Chapter 11

Further Study

11.1 INTRODUCTION

The scope of the research work, described so far, can be extended in several directions, some of which have been mentioned in earlier chapters. By confining our attention to linear finite-dimensional systems, it has been possible to give a fairly complete account of the synthesis and design of observers for system state reconstruction and control. In principle, the two most obvious extensions are to the construction of observers for non-linear systems and infinite-dimensional systems.

For non-linear systems, it is recollected that the linearization procedure of Section 1.2 is only valid for small perturbations in the state and the control vectors about a steady-state operating point. Section 11.2 sketches a theory of observers for general non-linear systems. The synthesis of observers for a more structured special class of non-linear systems, the class of bi-linear systems, is discussed in Section 11.3. Examples of bi-linear systems include nuclear reactor systems, heat exchanger systems and biological systems.

Finite-dimensionality is violated when one studies the problem of reconstruction of the state of distributed parameter models. Such models arise in transmission networks, chemical processes, economic systems, etc. Distributed parameter systems are modelled in an infinite-dimensional state–space by sets of partial differential equations, or delay-differential equations, or integro-differential equations. It is often impossible to adequately characterize the dynamic behaviour of these systems by a finite set of coupled ordinary differential equations. Section 11.4 sketches a theory of state reconstruction for a particular class of infinite-dimensional systems, namely, delay-differential linear systems.

Finally, in Section 11.5, we list some representative engineering applications of observers in the state reconstruction and control of linear and non-linear systems for a variety of different problems. Let no one think, however, that we

have said the last word on observers, even for finite-dimensional linear systems—we haven't!

One rapidly developing field of investigation is that of multi-dimensional linear system theory [B14], in particular, the analysis and design of two-dimensional (2-D) digital systems involving signals which depend on more than one independent variable. Unlike 1-D systems that we have studied at length, the stability analysis of 2-D systems is rather more difficult. An observer theory for such systems is, however, beginning to emerge [S17].

Turning to large-scale systems, a decentralized approach to state reconstruction has been introduced by Aoki and Li [A11], Siljak and Vukcevic [S11] and Sundareshan [S20]. Essentially, observers designed on the basis of subsystem dynamics alone are modified so as to take account of subsystem interconnections, such that the overall state reconstruction scheme is suitably convergent. By reducing the computational requirements of observer design, Arbel and Tse [A12] have, on the other hand, developed a centralized approach which is attractive for systems of high state-dimension and moderately few outputs.

The area of linear finite-dimensional systems, itself, is by no means exhausted. Recent extensions include the synthesis (design) of observers for singularly perturbed linear systems [O13] and for linear servomechanism problems [F7], [O16]. A quantification of the effects of finite wordlengths on system performance in the design of digital observer-based controllers is conducted by Rink and Chong [R4]. Otherwise, little has been reported on the numerical and computational aspects of observer-controller design.

More generally, the outlook for future research in observer theory and applications is a healthy one. As the modelling and analysis of physical systems become progressively more complex, the problems of estimating the missing system variables grow more acute. The success of any feedback control strategy intending to deploy these estimated variables will depend on the ability to synthesize an appropriate reconstruction scheme. The observer, in suitably modified form, will continue to play a dominant role in this reconstruction of missing system variables. In securing this role, it is expected that the lively interplay between observers, system theory, dynamic feedback compensation and other related areas, emphasized herein, will be further developed and exploited. For the author's part, it has never been other than a pleasure to participate in this challenging field of research and he hopes that you will join with him in turning current potentialities into reality.

11.2 OBSERVERS FOR NON-LINEAR SYSTEMS

Consider the non-linear time-invariant system

$$\dot{x}(t) = f(x(t)) \tag{11.1}$$

$$y(t) = h(x(t)). \tag{11.2}$$

The state reconstruction problem is one of synthesizing an observer, using $y(t) \in R^m$ as an input to the observer, in order to reconstruct the system state $x(t) \in R^n$. Restricting our attention to full-order observers, an appropriate observer would take the form

$$\dot{z} = f(z) + g[y, h(z)] \tag{11.3}$$

where $z(t) \in R^n$, and

$$g[y, h(z)] = 0 \quad \text{if } h(x) = h(z). \tag{11.4}$$

In the happy though unlikely event that the initial state of the system were known, one may set $z(t_0) \equiv x(t_0)$ so that $g = 0$ for all $t \geqslant t_0$, and the observer dynamics (11.3) reduces to exactly those of the system (11.1); hence we have $x(t) \equiv z(t)$ for all $t \geqslant t_0$ or perfect state reconstruction.

Generally, $x(t_0)$ is unknown and the most promising line of attack is to find a function g satisfying (11.4) such that, whatever the initial error $e(t_0) \triangleq x(t_0) - z(t_0)$, the error $e(t) \triangleq x(t) - z(t)$ at any time t is suitably "small". What we seek is exponential decay of the state reconstruction error $e(t)$; in other words, the exponential stability of the error dynamics, and this is provided by the application of a Lyapunov-like method to the composite system-observer (11.1) and (11.3).

Support we consider the case where the function g of (11.3), satisfying (11.4), is a linear function of $h(x) - h(z)$, say

$$g[y, h(z)] = B[y - h(z)] = B[h(x) - h(z)] \tag{11.5}$$

where B is a constant $n \times n$ matrix. The observer (11.3) becomes

$$\dot{z} = f(z) + B[h(x) - h(z)]. \tag{11.6}$$

Then, by pursuing a Lyapunov-type approach, we have the following result due to Kou *et al.* [K28].

Theorem 11.1 *If there exists a constant $n \times m$ matrix B and a positive definite, symmetric $n \times n$ matrix Q such that*

$$Q(\nabla f(x) - B\nabla h(x)) \tag{11.7}$$

is uniformly negative definite, then the dynamic system (11.6), with the matrix B satisfying (11.7) and with any $z(t_0)$, is an exponential observer for the system (11.1) and (11.2) and

$$\|z(t) - x(t)\| \leqslant \alpha_1 \|z(t_0) - x(t_0)\| \exp[-\alpha_2(t - t_0)] \tag{11.8}$$

for all $t \geqslant t_0$ and some positive numbers α_1 and α_2.

Two points associated with the application of the so-called direct stability method of Lyapunov are worth noting. First, Theorem 11.1 provides sufficient conditions for an exponential observer and these conditions tend to be conservative. Secondly, satisfaction of Equation (11.7) does not of itself constitute a constructive procedure for determining a stabilizing gain matrix B. The choice of B in order to satisfy (11.7) is a trial-and-error process that may be well-nigh impossible for systems of high order. These limitations are part of the difficulty of analysing general non-linear systems.

11.3 OBSERVERS FOR BI-LINEAR SYSTEMS

A special class of non-linear systems possessing many of the properties of linear systems is the class of bi-linear systems:

$$\dot{x} = A^0 x + \sum_{i=1}^{r} A^i u_i x + Bu; \quad x(0) = x_0 \tag{11.9}$$

$$y = Cx \tag{11.10}$$

where $x \in R^n$, $u = [u_1, u_2, \ldots, u_r]' \in R^r$, and $y \in R^m$. Unlike a linear system, however, the observability of the bi-linear system (11.9) and (11.10) is affected by the input $u \in R^r$.

A candidate observer that will reconstruct a linear function of the state, say Kx, is represented by

$$\dot{z} = D^0 z + \sum_{i=1}^{r} D^i u_i z + E^0 y + \sum_{i=1}^{r} E^i u_i y + Gu; \quad z(0) = z_0 \tag{11.11}$$

$$w = Pz + Vy \tag{11.12}$$

where $z \in R^q$, $w \in R^p$, and (D^0, P) is an observable pair. Guided by the results of linear observer theory in Chapter 3, it is required that the estimation error be independent of $u(\cdot)$, x_0 and z_0. In line with Definition 3.1, the dynamic system (11.11) and (11.12) is said to be a q-dimensional observer for Kx if and only if

$$\lim_{t \to \infty} [w(t) - Kx(t)] = 0 \tag{11.13}$$

independent of $u(\cdot)$, x_0 and z_0.

Similar to Theorem 3.2, necessary and sufficient conditions for the system (11.11) and (11.12) to constitute a linear function observer in the sense of (11.13) are supplied by the following theorem.

Theorem 11.2 *The dynamic system (11.11) and (11.12) is a q-dimensional observer for Kx if and only if there exists a $q \times n$ matrix T satisfying the following*

conditions:

$$D^0 \text{ is a stability matrix} \tag{11.14}$$

$$TA^0 - D^0T = E^0C \tag{11.15}$$

$$G = TB \tag{11.16}$$

$$K = PT + VC \tag{11.17}$$

$$D^i = 0, \quad i = 1, \dots, r \tag{11.18}$$

$$TA^i = E^iC, \quad i = 1, \dots, r. \tag{11.19}$$

Conditions (11.14)–(11.17) are identical to those of Theorem 3.2 for a linear system. The extra conditions (11.18) and (11.19) ensure that the input $u(\cdot)$, reflected in the bi-linear terms of Equations (11.9) and (11.11), is not transmitted into the estimation error.

11.4 OBSERVERS FOR DELAY-DIFFERENTIAL SYSTEMS

Consider the time-invariant linear delay-differential equation

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + Bu(t); \quad t \geqslant 0 \tag{11.20}$$

with measurement equation

$$y(t) = Cx(t) \tag{11.21}$$

and initial condition

$$x(t) = g(t), \quad -\tau \leqslant t \leqslant 0 \tag{11.22}$$

where τ is some positive constant and $g(t)$ is a continuous function on the interval $[-\tau, 0]$. Thus, the state–space is the infinite-dimensional space of continuous functions on $[-\tau, 0]$.

It is required to reconstruct the system state vector $x(t) \in R^n$ for all $t > 0$ through the use of a minimal-order state observer described by the delay-differential equation for $t > 0$

$$\dot{z}(t) = TA_0Pz(t) + TA_1Pz(t-\tau) + TA_0Vy(t) + TA_1Vy(t-\tau) + TBu(t) \tag{11.23}$$

with initial condition

$$z(t) = h(t), \quad -\tau \leqslant t \leqslant 0 \tag{11.24}$$

where $h(t) \in R^{n-m}$ is a continuous function on $[-\tau, 0]$. The state estimate

$\hat{x}(t) \in R^n$ is defined for all $t \geqslant -\tau$ by the equation

$$\hat{x}(t) = Pz(t) + Vy(t). \tag{11.25}$$

Theorem 11.3 *Assume, without loss of generality, that C is of full rank m, and T is an $(n - m) \times m$ matrix. If the matrices P and V satisfy the equation*

$$PT + VC = I_n \tag{11.26}$$

then the state reconstruction error vector $e(t) \triangleq x(t) - \hat{x}(t)$ satisfies the delay-differential equation for $t > 0$

$$\dot{e}(t) = PTA_0e(t) + PTA_1e(t - \tau) \tag{11.27}$$

with initial conditions

$$e(t) = Ph(t) + VCg(t), \quad -\tau \leqslant t \leqslant 0.$$

Furthermore, if $e(t) = 0$ on $-\tau \leqslant t \leqslant 0$, then $\hat{x}(t) = x(t)$ for all $t > 0$.

Proof Since $e(t)$ is differentiable for $t > 0$, the derivative satisfies the delay-differential equation

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= (I - VC)A_0x + (I - VC)A_1x(t - \tau) - PTA_0Vy \\ &\quad - PTA_1Vy(t - \tau) + [I - PT - VC]Bu. \end{aligned} \tag{11.28}$$

Equation (11.27) readily follows by substitution of (11.25) and (11.26) in (11.28). The final statement follows from uniqueness of solution of linear differential equations. Q.E.D. □

When $\tau = 0$, we have the familiar observer result of Section 1.4 for non-delayed systems. A sufficient condition for the asymptotic stability of Equation (11.27), in the sense of Krasovskii [H6], is that every root of the equation

$$\det(PTA_0 + PTA\exp(-\lambda\tau) - \lambda I) = 0 \tag{11.29}$$

is situated in the half-plane $\operatorname{Re}\lambda \leqslant \alpha < 0$.

Since Equation (11.29) has a countably infinite number of roots for constant τ, the determination of stability is much more complicated than for the non-delay case. It is readily deduced from Section 2.4 that if (A_0, C) or (A_1, C) is an observable pair, the eigenvalues of PTA_0 or PTA_1 can be arbitrarily assigned. In general, however, it may not be possible to choose P and T to satisfy (11.29).

For *small* delay systems, by assuming observability of (11.20) for $\tau = 0$, one can estimate a region of stability for (11.27).

Theorem 11.4 *If the pair $(A_0 + A_1, C)$ is observable, then there exist observer matrices P and T such that (11.27) is asymptotically stable whenever $\tau = 0$.*

Moreover, for every τ in the interval

$$0 \leqslant \tau \leqslant \frac{\alpha}{\beta |PTA_0||PTA_1|} \tag{11.30}$$

Equation (11.27) is asymptotically stable where α and β are constants.

11.5 SOME ENGINEERING APPLICATIONS

The observer theory established herein is an essential starting point in the systematic resolution of practical synthesis and design problems which require the use of state reconstruction. Engineers will be especially aware that the theory reflects an idealization that must be squared with an ignorance of the system model, imprecise performance specifications and extraneous economic constraints that go with actual application. Many readers with specific applications in mind will doubtless be *au courant* of the feasibility of observers in their chosen area of interest; the more so in the light of the recent spectacular advances in inexpensive digital computer technology. Nonetheless, we take the opportunity to mention some representative applications of observers for both linear and non-linear systems.

One of the first reported applications of observers in an industrial environment was to a computer-controlled pilot plant evaporator [S6]. Experimental evaluation revealed that a deterministic observer performed well under normal conditions but was sensitive to process noise and unmeasured process disturbances. This is consistent with the findings of Section 4.3 and Section 10.4, and bears out the importance of choosing the proper observer for the appropriate system model—a stochastic observer-estimator for an inherently stochastic system, and so on. Mitchell and Harrison [M14] design a hardware observer for the active control of a machine tool such as a lathe in order to reduce clatter tendency and forced vibration effects. A procedure is presented by Frank and Keller [F8] for the design of sensitivity discriminating observers for the detection of instrument failures.

In aircraft flight control, it is shown [S9] that a state observer can be used to reconstruct sensor signals, in the event of sensor failures, from the information extracted from aircraft altitude sensors. A linear observer is designed by Singh and Schy [S14] to construct the difficult-to-measure angle of attack and side-slip of a rapidly manoeuvring aircraft.

Okungwu *et al.* [O2] consider the excitation control of a microalternator using linear state feedback for which the inaccessible states are reconstructed by an observer. A non-linear state observer is devised by Dote [D8] to reconstruct the unmeasurable torque or flux of an induction motor. This

enables the stabilization of an unstable induction motor with a limit cycle.

Williamson [W7] has discovered that, although biological sensors are not available for on-line control of microbial cell growth in waste treatment and fermentation systems, an observable bilinear model permits the use of an on-line observer to estimate the missing variables.

Finally, a linear state observer is designed by McLane and Peppard [M8] to estimate the carriage velocities of a multi-locomotive powered train.

11.6 NOTES AND REFERENCES

The brief treatment of observers for non-linear systems in Section 11.2 follows Kou *et al.* [K28]; see also Banks [B2]. As a potentially useful alternative to non-linear filters, the Lyapunov-like method of observer design may be extended to non-linear stochastic systems [T1]. A second-order observer is obtained by Tsuji *et al.* [T8] through applying linear observer theory to an augmented linearized model in which each quadratic term of the original system is replaced by a new state variable. Our discussion of observers for bilinear systems is patterned on Hara and Furuta [H4]; see also Williamson [W7], Dote [D8] and Derese and Noldus [D6].

The introductory treatment of observers for delay-differential systems follows the original by Hewer and Nazaroff [H6]; more recent extensions include the contributions by Bhat and Koiva [B8], Hamidi-Hashemi and Leondes [H2], Salamon [S3] and Ogunnaike [O1]. A direct generalization of finite-dimensional linear observer theory to abstract linear systems characterized by semi-groups on Banach spaces is made by Gressang and Lamont [G6]; see also Kobayashi and Hitotsuya [K27]. An observer theory for general distributed-parameter linear systems is, however, far from complete.

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Appendix A

Some Matrix Theory

This appendix and Appendix B summarize some matrix and probabilistic results that are used in the text. Of the many good texts on matrix theory and linear algebra, currently available, the most comprehensive treatment that continues to inspire research in linear observer and system theory is that of Gantmacher [G1]; see also Bellman [B7] and MacDuffee [M1]. For an annotated list of several other books on matrix theory, linear algebra and numerical analysis see Kailath [K4].

A.1 MATRICES

An $m \times n$ *matrix* is a rectangular array of numbers, called *elements*, arranged in m rows and n columns.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The element in the ith row and the jth column is denoted by a_{ij}. A matrix is *rectangular* if $m \neq n$ and *square* if $m = n$. The identity matrix, I_n, is a square $n \times n$ matrix with elements $a_{ij} = 0$ for $i \neq j$, and $a_{ii} = 1$ for $i = 1, \ldots, n$.

The *sum* of two $m \times n$ matrices A and B is defined by the expression

$$C = A + B = [a_{ij} + b_{ij}].$$

The *product* of an $m \times n$ matrix A and an $n \times p$ matrix B is the $m \times p$ matrix C defined by

$$C = AB = [a_{ij}][b_{ij}] = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right] = [c_{ij}].$$

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A' with elements $a'_{ij} = a_{ji}$. A square matrix A is *symmetric* if $A' = A$.

Determinant and matrix inverse

It is recalled from elementary algebra that there is associated with each square $n \times n$ matrix A a unique number known as the *determinant* of A. It is denoted by det A and is commonly evaluated by using the *Laplace expansion*

$$\det A = \sum_{j=1}^{n} a_{ij}\gamma_{ij} \quad \text{for any } i = 1, 2, \dots, n, \quad \text{with } \gamma_{ij} = (-1)^{i+j}\mu_{ij}$$

where μ_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the row and column in A in which a_{ij} is located. The scalars γ_{ij} and μ_{ij} are called, respectively, the *cofactor* and *minor* of a_{ij}. The $n \times n$ matrix $[\gamma_{ji}]$, that is, the transpose of the matrix $[\gamma_{ij}]$, is termed the *adjoint* of A, and is denoted by adj. A.

The evaluation of det A is often assisted by the following *elementary* (*row or column*) *operations.*

(1) If any column (or row) of A is multiplied by a scalar α and the resulting matrix is denoted by $\bar{A}$, then $\det \bar{A} = \alpha \det A$.
(2) If $\bar{A}$ is the matrix obtained from A by interchanging any two rows (or columns) of A, then $\det \bar{A} = -\det A$.
(3) If $\bar{A}$ is obtained from A by adding a multiple of any one row (or column) to another, then $\det \bar{A} = \det A$.
(4) $\det A' = \det A$.
(5) If A and B are any two square matrices, then $\det AB = \det BA$.

A square $n \times n$ matrix is *non-singular* if and only if the n vectors that constitute its rows (or columns) are linearly independent or, equivalently, if and only if $\det A \neq 0$.

The *inverse* of a square non-singular matrix A, denoted by A^{-1}, is

$$A^{-1} = \operatorname{adj} A / \det A.$$

If $\det A = 0$, the matrix A is singular and the matrix inverse A^{-1} does not exist. Instead, a type of approximate inverse called the *pseudo* or *generalized* inverse of A can be defined and is denoted by $A\dagger$ [B15].

A.2 VECTORS AND INNER PRODUCTS

The single column matrix $x = (x_1, \dots, x_n)$ is known as the n-dimensional *column vector* x. If $x_1, x_2, \dots, x_n$ are real (complex) numbers, x *is an element of an* n-dimensional vector space over a *field* of real numbers $\mathscr{R}$ (complex

numbers $\mathbb{C}$). It is in general assumed throughout the text that we are dealing with vectors over the real field $\mathscr{R}$, that is, real-valued vectors.

As a generalization of the notions of distance and angle in two- and three-dimensional spaces over the real field $\mathscr{R}$, we have the concept of *inner product* (or scalar product). A vector space together with an inner product is called an *inner-product space*. In particular, the n-dimensional inner-product space over the real field is the *Euclidean space*, denoted by R^n. The inner product of two column vectors $x \in R^n$ and $y \in R^n$ is defined by

$$x'y = \sum_{i=1}^{n} x_i y_i.$$

Two vectors $x \in R^n$ and $y \in R^n$ are said to be *orthogonal* if

$$x'y = \sum_{i=1}^{n} x_i y_i = 0.$$

The length of a vector $x \in R^n$ is described by the *Euclidean norm*, denoted by $\|x\|$, and is equal to

$$(x'x)^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

If $x \in R^n$ and $y \in R^n$, we have the *Cauchy–Schwartz inequality*

$$|x'y|^2 = \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqslant \|x\|^2 \|y\|^2$$

where the equality signs hold if and only if x and y are linearly dependent (that is, one is a scalar multiple of the other).

A.3 LINEAR INDEPENDENCE AND RANK

A set of vectors $x^1, x^2, \ldots, x^n$ is said to be *linearly independent* if there exists a set of scalars $\alpha_1, \ldots, \alpha_n$, at least one of which is non-zero, such that

$$\sum_{i=1}^{n} \alpha_i x^i = 0.$$

If no such set of scalars exists, the vectors are *linearly dependent*.

In vector-space language, we describe a *mapping* of vector space X into vector space Y that preserves the operations of addition and multiplication for all $x \in X$ and $y \in Y$, as a *linear transformation*; for example, $A: X \rightarrow Y$ is a linear transformation. The set of linearly independent vectors $\{x^1, \ldots, x^n\}$ which spans the vector space X acts as a *basis* for X. The array $[a_{ij}]$ is a matrix

representation of the linear transformation A with respect to the bases $\{x^1, \dots, x^n\}$ and $\{y^1, \dots, y^m\}$. In other words, we have the system of linear equations:

$$Ax = y.$$

This system of linear equations possesses a solution for the vector x if and only if the vector y is some linear combination of the columns of the matrix A. In this case, the equations are said to be *consistent* and y is said to lie in the *range space* (or *image space*) of A, denoted by range A (or Im A).

The *null space* or *kernel* of A, denoted by Ker A, is the space of all solutions of the equation

$$Ax = 0.$$

The *column rank* (*row rank*) of a matrix is equal to the number of linearly independent columns (rows). Any matrix whose columns are linearly independent is said to be of *full column rank.*

For any matrix A, column rank A = row rank A = rank A.

For any linear transformation $A : X \to Y$ dimension (kernel A) + dimension (range A) = dimension X, that is,

$$\text{nullity } A + \text{rank } A = \text{dimension } X.$$

A useful inequality on rank is *Sylvester's inequality* which states that if A is $m \times n$ and B is $n \times p$,

$$\text{rank } A + \text{rank } B - n \leqslant \text{rank } (AB) \leqslant \min \{\text{rank } A, \text{rank } B].$$

A.4 PARTITIONED MATRICES

It is often convenient to partition a matrix into sub-matrices. For example, a matrix may be partitioned as the block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Fortunately, the ordinary rules of matrix addition, multiplication etc., still apply to partitioned matrices.

Multiplication of partitioned matrices

Provided that the dimensions of the partitioned matrices are compatible, we have that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}.$$

Determinants of partitioned matrices

If A is a non-singular matrix,

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det\,[B - CA^{-1}D].$$

A useful identity for determinants of products is

$$\det\,[I_n - AB] = \det\,[I_m - BA]$$

where A is $n \times m$ and B is $m \times n$.

Inverse of partitioned matrices

If A and $\Delta \triangleq B - CA^{-1}D$ are non-singular matrices,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}D\Delta^{-1}CA^{-1} & -A^{-1}D\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix}.$$

Δ is known as the *Schur complement* of A.

If A and C are non-singular matrices.

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}.$$

A.5 EIGENVALUES, EIGENVECTORS AND CHARACTERISTIC POLYNOMIALS

A non-zero vector $v \in V$ is said to be an *eigenvector* of the linear transformation $A: V \rightarrow V$ if

$$Av = \lambda v$$

for some scalar λ. The scalar λ is called an *eigenvalue* of A. Since any subspace W of V is mapped into itself by A, W is said to be an *invariant subspace* of V under A. By rearranging terms, we seek a non-zero solution of the homogeneous system of equations

$$(\lambda I - A)v = 0.$$

A necessary and sufficient condition for this eigenvalue problem to have a non-trivial solution is that

$$\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0.$$

The n roots of this equation, known as the *characteristic equation* of the square $n \times n$ matrix A, are the n (not necessarily distinct) eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, of A. We sometimes refer to the set of eigenvalues of A as the *spectrum* of A. The left-hand side of the equation is called the *characteristic polynomial* of A, and it

may be written as

$$p(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

The eigenvalues of a real-element matrix are real or occur in complex-conjugate pairs.

Determinant and trace

It can be shown that

$$\det A = \prod_{i=1}^{n} \lambda_i$$

and

$$\operatorname{trace} A \triangleq \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i.$$

Also,

$$\operatorname{trace}(A + B) = \operatorname{trace} A + \operatorname{trace} B$$

$$\operatorname{trace}(ABC) = \operatorname{trace}(BCA) = \operatorname{trace}(CAB).$$

The Cayley–Hamilton theorem

The Cayley–Hamilton theorem states that every square matrix satisfies its own characteristic equation. That is, if A is an $n \times n$ square matrix with characteristic polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$, then

$$p(A) \triangleq A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0.$$

The expression on the left-hand side of this equation is a *polynomial in the matrix A*, or a *matrix polynomial.*

The minimal polynomial

While the Cayley–Hamilton theorem states that $p(A) = 0$, there may be polynomials $\delta(s)$ of lower degree than $p(s)$ such that $\delta(A) = 0$. The monic (or unity leading coefficient) polynomial of lowest degree for which $\mu(A) = 0$ is called the *minimum polynomial* of A.

Nilpotent matrix

A square $n \times n$ matrix A is said to be *nilpotent* if $A^r = 0$ for some $r > 1$; r is known as the *index of nilpotency* of A. All the eigenvalues of a nilpotent matrix are zero.

Resolvent matrix

The matrix $(sI - A)^{-1}$ is known as the *resolvent* of A and satisfies the power

expansion

$$(sI - A)^{-1} = s^{-1} + As^{-2} + A^2s^{-3} + \cdots.$$

Positive-definite matrices

The square symmetric matrix A is *positive definite* (*positive semi-definite*) if the quadratic form $x'Ax > 0$ ($x'Ax \geqslant 0$) for all x. A symmetric matrix A is positive definite (positive semi-definite) if and only if all its eigenvalues are positive (non-negative).

A.6 INNER PRODUCTS, TRACE FUNCTIONS AND GRADIENT MATRICES

Analogously to the inner product of two vectors in Section A.2, one may define an *inner product* between two matrices. If A and B are $n \times n$ matrices, over the real field $\mathscr{R}$, i.e. $A \in R^{nm}$, $B \in R^{nm}$, their inner product is defined by the trace operation,

$$(A, B) = \text{trace}\,[AB'] = \sum_{i=1}^{n} \sum_{i=1}^{m} a_{ij}b_{ij}.$$

Properties of the inner product are

$$\text{tr}\,[AB'] = \text{tr}\,[BA']$$
$$\text{tr}\,[\alpha AB'] = \alpha\,\text{tr}\,[AB'] \quad \text{for any scalar } \alpha$$
$$\text{tr}\,[(A + B)C'] = \text{tr}\,[AC'] + \text{tr}\,[BC'].$$

A *gradient matrix* is defined as follows: suppose X is an $n \times n$ real-element matrix. Let $f(X)$ be a scalar function of the elements x_{ij} of X; that is,

$$f(X) = f(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{nn}).$$

Then, the gradient matrix of $f(X)$ is denoted by

$$\frac{\partial f(X)}{\partial X}$$

and it is an $n \times n$ matrix whose ijth element is given by

$$\left[\frac{\partial f(X)}{\partial X}\right]_{ij} = \frac{\partial f}{\partial x_{ij}}.$$

Some common gradient matrices include the following:

$$\frac{\partial}{\partial X}\,\text{tr}\,[X] = I$$

$$\frac{\partial}{\partial X}\operatorname{tr}[AX] = A'$$

$$\frac{\partial}{\partial X}\operatorname{tr}[AX'] = A$$

$$\frac{\partial}{\partial X}\operatorname{tr}[AXB] = A'B'$$

$$\frac{\partial}{\partial X}\operatorname{tr}[AX'B] = BA.$$

Further results and applications of gradient matrices are contained in Athans [A13].

Appendix B

A Little Probability Theory

Thie appendix introduces some elementary probabilistic notions used in Chapter 6. More detailed treatments include those by Feller [F5], Papoulis [P2] and Parzen [P4].

Probability space

An *event* is defined as some specific class of outcomes of an experiment in which chance is involved. Inthis connection, an event A is said to occur if and only if the observed outcome of the experiment is an element of A. The set of all possible outcomes of such an experiment is called the *sample space*, denoted by Ω, and containing elements ω.

Now suppose that the experiment is performed N times and that in these N trials, an event occurs $N(A)$ times. Then we say that the probability of the event A, denoted by $P(A)$, is defined by the relation

$$P(A) = \lim_{N \to \infty} \frac{N(A)}{N}, \qquad 0 \leqslant N(A) \leqslant N$$

assuming the limit exists.

Put more formally, a *probability space* $(\Omega, \mathscr{F}, P)$ is defined as follows:

A probability space $(\Omega, \mathscr{F}, P)$ consists of

(a) a sample space Ω of elements ω;

(b) a collection (or *Borel field*) $\mathscr{F}$ of subsets of Ω, called events, which includes Ω and has the following properties:

(1) If A is an event, then the complement $\bar{A} = \{\omega \in \Omega \mid \omega \notin A\}$ is also an event. (The complement of Ω is the empty set and is considered to be an event.)

(2) If A_1, A_2 are events, then $A_1 \cap A_2$, $A_1 \cup A_2$ are also events.

(3) If $A_1, A_2, \ldots, A_k, \ldots$ are events, then $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$ are also events.

(c) a function $P(\cdot)$ assigning to each event A a real number $P(A)$, called the *probability of the event* A, and satisfying

(1) $P(A) \geqslant 0$ for every event A.
(2) $P(\Omega) = 1$.
(3) $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ for every pair of disjoint events A_1, A_2.
(4) $P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$ for every sequence of mutually disjoint events $A_1, A_2, \dots, A_k, \dots$.

The function P is referred to as a *probability measure.*

Random variables, probability functions and expected values

Given a probability space $(\Omega, \mathscr{F}, P)$, a *random variable* is a real-valued function $x(\omega)$ on Ω such that the set $\{\omega \in \Omega \mid x(\omega) \leqslant \lambda\}$, λ real, is an event, i.e. belongs to the collection $\mathscr{F}$.

An n-dimensional *random vector* $x = (x_1, x_2, \dots, x_n)$ is an n-tuple of random variables $x_1, x_2, \dots, x_n$ each defined on the same probability space.

The *probability distribution function* of a random vector $x = (x_1, x_2, \dots, x_n)$ is defined by

$$F(z_1, z_2, \dots, z_n) = P(\{\omega \in \Omega \mid x_1(\omega) \leqslant z_1, x_2(\omega) \leqslant z_2, \dots, x_n(\omega) \leqslant z_n\})$$

that is, $F(z)$ is equal to the probability that the elements of the random vector x take values less than or equal to those of the vector z.

The random variables $x_1, \dots, x_n$ are said to be *independent* if

$$F(z_1, \dots, z_n) = F(z_1) \cdot F(z_2) \cdots F(z_n)$$

for all scalars $z_1, \dots, z_n$.

The *expected value* of a random vector x with distribution function F is defined as

$$E[x] = \int_{-\infty}^{\infty} z dF(z)$$

provided the above integral is well defined.

The *covariance matrix* of a random column vector x with expected value $E[x]$ is defined to be the $n \times n$ symmetric positive semi-definite matrix

$$E[(x - E[x])(x - E[x])'] = Q.$$

Two random column vectors x and y are said to be *uncorrelated* if

$$E[(x - E[x])(y - E[y])'] = 0.$$

The random vector $x = (x_1, \dots, x_n)$ is said to be characterized by a piecewise continuous *probability density function* f if f is piecewise continuous and

$$F(z_1, \dots, z_n) = \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \cdots \int_{-\infty}^{z_n} f(y_1, \dots, y_n)\, dy_1 \cdots dy_n$$

for every $z_1, \dots, z_n$.

Conditional probability

Consider two random vectors x and y taking values in R^n and R^m, respectively. We define the *conditional probability* of x given y by

$$P(X \mid Y) = P(\omega \mid x(\omega) \in X\} \mid \{\omega \mid y(\omega) \in Y\})$$

where X and Y are subsets of R^n and R^m, respectively.

For a fixed vector $\omega \in R^n$ we define the *conditional distribution function* of x given w by

$$F(z \mid w) = P(\{\omega \mid x(\omega) \leqslant z\} \mid \{\omega \mid y(\omega) = w\})$$

and the *conditional expectation* of x given w by

$$E[x \mid w] = \int_{R^n} z dF(z \mid w)$$

provided that the above integral is well defined. Note that the conditional expectation $E[x \mid w]$ is itself a random vector. Similarly, one may define the conditional covariance of the random vector x given w, etc.

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